AUTHORIZED

TEXT BOOK SERIES

# TODHUNTER'S EUCLID

FOR

SCHOOLS & COLLEGES

BOOKS I., II., III.

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# THE ELEMENTS OF

# EUCLID

FOR THE USE OF SCHOOLS AND COLLEGES.

## BOOKS I., II., III.

BY

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Authorized by the Education Department of Ontario.

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# EUCLID'S ELEMENTS.

## BOOK I.

### DEFINITIONS.

1. A POINT is that which has no parts, or which has no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

5. A superficies is that which has only length and breadth.

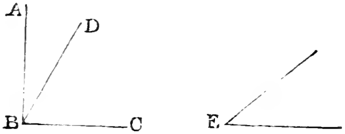
6. The extremities of a superficies are lines.

7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

*Note.* When several angles are at one point $B$, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle, that is, at the point at which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus, the angle which is contained by the

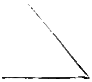



straight lines $AB$, $CB$ is named the angle $ABC$, or $CBA$; the angle which is contained by the straight lines $AB$, $DB$ is named the angle $ABD$, or $DBA$; and the angle which is contained by the straight lines $DB$, $CB$ is named the angle $DBC$, or $CBD$; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at $E$.

10. When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

11. An obtuse angle is that which is greater than a right angle.

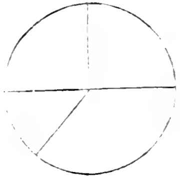
12. An acute angle is that which is less than a right angle.



**13. A term or boundary is the extremity of any thing.**

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another:



16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

[A radius of a circle is a straight line drawn from the centre to the circumference.]

18. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A segment of a circle is the figure contained by a straight line and the circumference which it cuts off.

20. Rectilineal figures are those which are contained by straight lines:

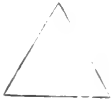
21. Trilateral figures, or triangles, by three straight lines:

22. Quadrilateral figures by four straight lines:

23. Multilateral figures, or polygons, by more than four straight lines.

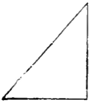
24. Of three-sided figures,

An equilateral triangle is that which has three equal sides:

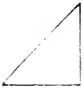


25. An isosceles triangle is that which has two sides equal:

26. A scalene triangle is that which has three unequal sides:



27. A right-angled triangle is that which has a right angle:



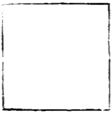
[The side opposite to the right angle in a right-angled triangle is frequently called the hypotenuse.]

28. An obtuse-angled triangle is that which has an obtuse angle:

29. An acute-angled triangle is that which has three acute angles.

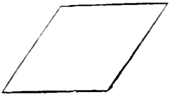
Of four-sided figures,

30. A square is that which has all its sides equal, and all its angles right angles:

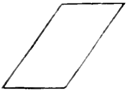


31. An oblong is that which has all its angles right angles, but not all its sides equal:

32. A rhombus is that which has all its sides equal, but its angles are not right angles:



33. A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles:



34. All other four-sided figures besides these are called trapeziums.

35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

[*Note.* The terms *oblong* and *rhomboid* are not often used. Practically the following definitions are used. Any four-sided figure is called a *quadrilateral*. A line joining two opposite angles of a quadrilateral is called a *diagonal*. A quadrilateral which has its opposite sides parallel is called a *parallelogram*. The words *square* and *rhombus* are used in the sense defined by Euclid; and the word *rectangle* is used instead of the word *oblong*.

Some writers propose to restrict the word *trapezium* to a quadrilateral which has two of its sides parallel; and it would certainly be convenient if this restriction were universally adopted.]

## POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any centre, at any distance from that centre.

## AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

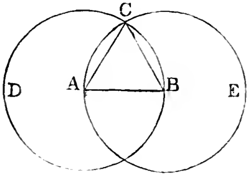
11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

## PROPOSITION 1. *PROBLEM.*

*To describe an equilateral triangle on a given finite straight line.*

Let $AB$ be the given straight line: it is required to describe an equilateral triangle on $AB$.




From the centre $A$, at the distance $AB$, describe the circle $BCD$. [*Postulate* 3.

From the centre $B$, at the distance $BA$, describe the circle $ACE$. [*Postulate* 3.

From the point $C$, at which the circles cut one another, draw the straight lines $CA$ and $CB$ to the points $A$ and $B$. [*Post.* 1.

$ABC$ shall be an equilateral triangle.

Because the point $A$ is the centre of the circle $BCD$, $AC$ is equal to $AB$. [*Definition* 15.

And because the point $B$ is the centre of the circle $ACE$, $BC$ is equal to $BA$. [*Definition* 15.

But it has been shewn that $CA$ is equal to $AB$;

therefore $CA$ and $CB$ are each of them equal to $AB$.

But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore $CA$ is equal to $CB$.

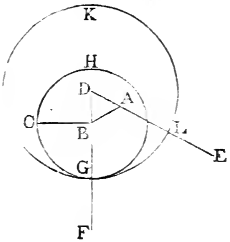
Therefore $CA$, $AB$, $BC$ are equal to one another.

Wherefore *the triangle $ABC$ is equilateral,* [*Def.* 24.
*and it is described on the given straight line $AB$.* Q.E.F.

PROPOSITION 2. *PROBLEM.*

*From a given point to draw a straight line equal to a given straight line.*

Let $A$ be the given point, and $BC$ the given straight line: it is required to draw from the point $A$ a straight line equal to $BC$.




From the point $A$ to $B$ draw the straight line $AB$; [*Post.* 1.
and on it describe the equilateral triangle $DAB$, [I. 1.
and produce the straight lines $DA$, $DB$ to $E$ and $F$. [*Post.* 2.
From the centre $B$, at the distance $BC$, describe the circle $CGH$, meeting $DF$ at $G$. [*Post.* 3.
From the centre $D$, at the distance $DG$, describe the circle $GKL$, meeting $DE$ at $L$. [*Post.* 3.
$AL$ shall be equal to $BC$.

Because the point $B$ is the centre of the circle $CGH$, $BC$ is equal to $BG$. [*Definition* 15.
And because the point $D$ is the centre of the circle $GKL$, $DL$ is equal to $DG$; [*Definition* 15.
and $DA$, $DB$ parts of them are equal; [*Definition* 24.
therefore the remainder $AL$ is equal to the remainder $BG$. [*Axiom* 3.
But it has been shewn that $BC$ is equal to $BG$;
therefore $AL$ and $BC$ are each of them equal to $BG$.
But things which are equal to the same thing are equal to one another. [*Axiom* 1.
Therefore $AL$ is equal to $BC$.

Wherefore *from the given point $A$ a straight line $AL$ has been drawn equal to the given straight line $BC$.* Q.E.F.

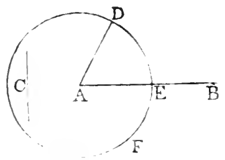
PROPOSITION 3. *PROBLEM.*

*From the greater of two given straight lines to cut off a part equal to the less*

Let $AB$ and $C$ be the two given straight lines, of which

$AB$ is the greater: it is required to cut off from $AB$, the greater, a part equal to $C$ the less.

From the point $A$ draw the straight line $AD$ equal to $C$; [I. 2.




and from the centre $A$, at the distance $AD$, describe the circle $DEF$ meeting $AB$ at $E$. [*Postulate* 3.

$AE$ shall be equal to $C$.

Because the point $A$ is the centre of the circle $DEF$, $AE$ is equal to $AD$. [*Definition* 15.

But $C$ is equal to $AD$. [*Construction.*

Therefore $AE$ and $C$ are each of them equal to $AD$.

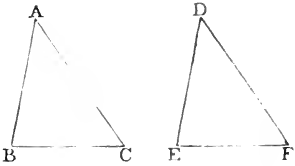
Therefore $AE$ is equal to $C$. [*Axiom* 1

Wherefore *from $AB$ the greater of two given straight lines a part $AE$ has been cut off equal to $C$ the less.* Q.E.F.

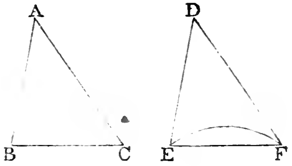
## PROPOSITION 4. *THEOREM.*

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.*

Let $ABC, DEF$ be two triangles which have the two sides $AB, AC$ equal to the two sides $DE, DF$, each to each, namely, $AB$ to $DE$, and $AC$ to $DF$, and the angle $BAC$ equal to the angle $EDF$: the base $BC$ shall be equal to the base $EF$, and the triangle $ABC$ to the triangle $DEF$, and the other angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle $ABC$ to the angle $DEF$, and the angle $ACB$ to the angle $DFE$.



For if the triangle $ABC$ be applied to the triangle $DEF$, so that the point $A$ may be on the point $D$, and the straight line $AB$ on the straight line $DE$, the point $B$ will coincide with the point $E$, because $AB$ is equal to $DE$. [*Hyp.*




And, $AB$ coinciding with $DE$, $AC$ will fall on $DF$, because the angle $BAC$ is equal to the angle $EDF$. [*Hypothesis.*

Therefore also the point $C$ will coincide with the point $F$, because $AC$ is equal to $DF$. [*Hypothesis.*

But the point $B$ was shewn to coincide with the point $E$, therefore the base $BC$ will coincide with the base $EF$;

because, $B$ coinciding with $E$ and $C$ with $F$, if the base $BC$ does not coincide with the base $EF$, two straight lines will enclose a space; which is impossible. [*Axiom* 10.

Therefore the base $BC$ coincides with the base $EF$, and is equal to it. [*Axiom* 8.

Therefore the whole triangle $ABC$ coincides with the whole triangle $DEF$, and is equal to it. [*Axiom* 8.

And the other angles of the one coincide with the other angles of the other, and are equal to them, namely, the angle $ABC$ to the angle $DEF$, and the angle $ACB$ to the angle $DFE$.

Wherefore, *if two triangles* &c. Q.E.D.

## PROPOSITION 5. *THEOREM.*

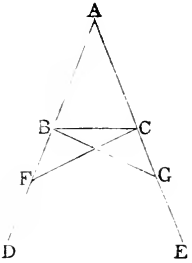
*The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.*

Let $ABC$ be an isosceles triangle, having the side $AB$ equal to the side $AC$, and let the straight lines $AB$, $AC$ be produced to $D$ and $E$: the angle $ABC$ shall be equal to the angle $ACB$, and the angle $CBD$ to the angle $BCE$.

In $BD$ take any point $F$,

and from $AE$ the greater cut off $AG$ equal to $AF$ the less, [I. 3.

and join $FC$, $GB$.




Because $AF$ is equal to $AG$, [*Constr.*
and $AB$ to $AC$, [*Hypothesis.*
the two sides $FA$, $AC$ are equal to the two sides $GA$, $AB$, each to each; and they contain the angle $FAG$ common to the two triangles $AFC$, $AGB$;
therefore the base $FC$ is equal to the base $GB$, and the triangle $AFC$ to the triangle $AGB$, and the remaining angles of the one to the remaining angles of the other, each to each, to which the equal sides are opposite, namely the angle $ACF$ to the angle $ABG$, and the angle $AFC$ to the angle $AGB$. [I. 4.

And because the whole $AF$ is equal to the whole $AG$, of which the parts $AB$, $AC$ are equal. [*Hypothesis.*
the remainder $BF$ is equal to the remainder $CG$. [*Axiom* 3.
And $FC$ was shewn to be equal to $GB$;
therefore the two sides $BF$, $FC$ are equal to the two sides $CG$, $GB$, each to each;
and the angle $BFC$ was shewn to be equal to the angle $CGB$;
therefore the triangles $BFC$, $CGB$ are equal, and their other angles are equal, each to each, to which the equal sides are opposite, namely the angle $FBC$ to the angle $GCB$, and the angle $BCF$ to the angle $CBG$. [I. 4.

And since it has been shewn that the whole angle $ABG$ is equal to the whole angle $ACF$,
and that the parts of these, the angles $CBG$, $BCF$ are also equal;
therefore the remaining angle $ABC$ is equal to the remaining angle $ACB$, which are the angles at the base of the triangle $ABC$. [*Axiom* 3.

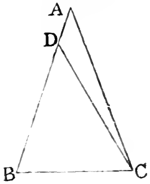
And it has also been shewn that the angle $FBC$ is equal to the angle $GCB$, which are the angles on the other side of the base.

Wherefore, *the angles* &c. Q.E.D.

Corollary. Hence every equilateral triangle is also equiangular.

PROPOSITION 6. *THEOREM.*

*If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.*




Let $ABC$ be a triangle, having the angle $ABC$ equal to the angle $ACB$: the side $AC$ shall be equal to the side $AB$.

For if $AC$ be not equal to $AB$, one of them must be greater than the other.

Let $AB$ be the greater, and from it cut off $DB$ equal to $AC$ the less, [I. 3.
and join $DC$.

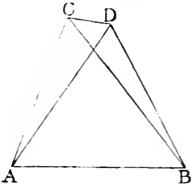
Then, because in the triangles $DBC$, $ACB$,
$DB$ is equal to $AC$, [*Construction.*
and $BC$ is common to both,
the two sides $DB$, $BC$ are equal to the two sides $AC$, $CB$, each to each;
and the angle $DBC$ is equal to the angle $ACB$; [*Hypothesis.*
therefore the base $DC$ is equal to the base $AB$, and the triangle $DBC$ is equal to the triangle $ACB$, [I. 4.
the less to the greater; which is absurd. [*Axiom* 9.
Therefore $AB$ is not unequal to $AC$, that is, it is equal to it.

Wherefore, *if two angles* &c. Q.E.D.

Corollary. Hence every equiangular triangle is also equilateral.

PROPOSITION 7. *THEOREM.*

*On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.*




If it be possible, on the same base $AB$, and on the same side of it, let there be two triangles $ACB$, $ADB$, having their sides $CA$, $DA$, which are terminated at the extremity $A$ of the base, equal

to one another, and likewise their sides $CB$, $DB$, which are terminated at $B$ equal to one another.

Join $CD$. In the case in which the vertex of each triangle is without the other triangle;

because $AC$ is equal to $AD$, [*Hypothesis.*

the angle $ACD$ is equal to the angle $ADC$. [I. 5.

But the angle $ACD$ is greater than the angle $BCD$, [*Ax.* 9.

therefore the angle $ADC$ is also greater than the angle $BCD$;

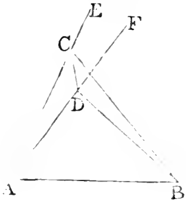
much more then is the angle $BDC$ greater than the angle $BCD$.

Again, because $BC$ is equal to $BD$, [*Hypothesis.*

the angle $BDC$ is equal to the angle $BCD$. [I. 5.

But it has been shewn to be greater; which is impossible.

But if one of the vertices as $D$, be within the other triangle $ACB$, produce $AC$, $AD$ to $E$, $F$.




Then because $AC$ is equal to $AD$, in the triangle $ACD$, [*Hyp.*

the angles $ECD$, $FDC$, on the other side of the base $CD$, are equal to one another. [I. 5.

But the angle $ECD$ is greater than the angle $BCD$, [*Axiom* 9.

therefore the angle $FDC$ is also greater than the angle $BCD$;

much more then is the angle $BDC$ greater than the angle $BCD$.

Again, because $BC$ is equal to $BD$, [*Hypothesis.*

the angle $BDC$ is equal to the angle $BCD$. [I. 5.

But it has been shewn to be greater; which is impossible.

The case in which the vertex of one triangle is on a side of the other needs no demonstration.

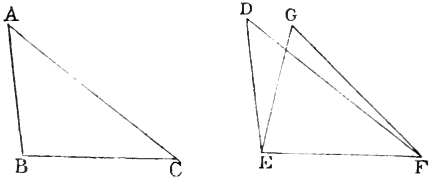
Wherefore, *on the same base* &c. Q.E.D.

PROPOSITION 8. *THEOREM.*

***If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their***

*bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.*

Let $ABC$, $DEF$ be two triangles, having the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$, each to each, namely $AB$ to $DE$, and $AC$ to $DF$, and also the base $BC$ equal to the base $EF$: the angle $BAC$ shall be equal to the angle $EDF$.



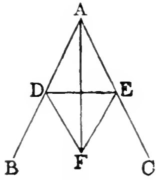

For if the triangle $ABC$ be applied to the triangle $DEF$, so that the point $B$ may be on the point $E$, and the straight line $BC$ on the straight line $EF$, the point $C$ will also coincide with the point $F$, because $BC$ is equal to $EF$. [*Hyp.*
Therefore, $BC$ coinciding with $EF$, $BA$ and $AC$ will coincide with $ED$ and $DF$.

For if the base $BC$ coincides with the base $EF$, but the sides $BA$, $CA$ do not coincide with the sides $ED$, $FD$, but have a different situation as $EG$, $FG$; then on the same base and on the same side of it there will be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise their sides which are terminated at the other extremity.
But this is impossible. [I. 7.
Therefore since the base $BC$ coincides with the base $EF$, the sides $BA$, $AC$ must coincide with the sides $ED$, $DF$.
Therefore also the angle $BAC$ coincides with the angle $EDF$, and is equal to it. [*Axiom* 8.

Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 9. *PROBLEM.*

*To bisect a given rectilineal angle, that is to divide it into two equal angles.*



Let $BAC$ be the given rectilineal angle: it is required to bisect it.

Take any point $D$ in $AB$, and from $AC$ cut off $AE$ equal to $AD$; [I. 3.

join $DE$, and on $DE$, on the side remote from $A$, describe the equilateral triangle $DEF$. [I. 1.

Join $AF$. The straight line $AF$ shall bisect the angle $BAC$.

Because $AD$ is equal to $AE$, [*Construction.*
and $AF$ is common to the two triangles $DAF$, $EAF$,
the two sides $DA$, $AF$ are equal to the two sides $EA$, $AF$, each to each;

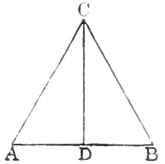
and the base $DF$ is equal to the base $EF$; [*Definition* 24.

therefore the angle $DAF$ is equal to the angle $EAF$. [I. 8.

Wherefore *the given rectilineal angle $BAC$ is bisected by the straight line $AF$.* Q.E.F.

PROPOSITION 10. *PROBLEM.*

*To bisect a given finite straight line, that is to divide it into two equal parts.*



Let $AB$ be the given straight line: it is required to divide it into two equal parts.

Describe on it an equilateral triangle $ABC$, [I. 1.

and bisect the angle $ACB$ by the straight line $CD$, meeting $AB$ at $D$. [I. 9.

$AB$ shall be cut into two equal parts at the point $D$.

Because $AC$ is equal to $CB$, [*Definition* 24.
and $CD$ is common to the two triangles $ACD$, $BCD$,
the two sides $AC$, $CD$ are equal to the two sides $BC$, $CD$ each to each;

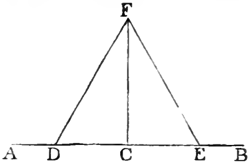
and the angle $ACD$ is equal to the angle $BCD$; [*Constr.*

therefore the base $AD$ is equal to the base $DB$. [I. 4.

Wherefore *the given straight line $AB$ is divided into two equal parts at the point $D$.* Q.E.F.

PROPOSITION 11. *PROBLEM.*

*To draw a straight line at right angles to a given straight line, from a given point in the same.*



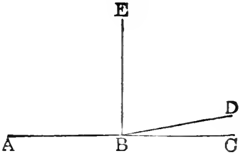
Let $AB$ be the given straight line, and $C$ the given point in it: it is required to draw from the point $C$ a straight line at right angles to $AB$.

Take any point $D$ in $AC$, and make $CE$ equal to $CD$. [I. 3
On $DE$ describe the equilateral triangle $DFE$, [I. 1.
and join $CF$.
The straight line $CF$ drawn from the given point $C$ shall be at right angles to the given straight line $AB$.

Because $DC$ is equal to $CE$, [*Construction.*
and $CF$ is common to the two triangles $DCF$, $ECF$;
the two sides $DC$, $CF$ are equal to the two sides $EC$, $CF$ each to each;
and the base $DF$ is equal to the base $EF$; [*Definition* 24.
therefore the angle $DCF$ is equal to the angle $ECF$; [I. 8.
and they are adjacent angles.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [*Definition* 10.
therefore each of the angles $DCF$, $ECF$ is a right angle.

Wherefore *from the given point $C$ in the given straight line $AB$, $CF$ has been drawn at right angles to $AB$.* Q.E.F.

Corollary. By the help of this problem it may be shewn that two straight lines cannot have a common segment.



If it be possible, let the two straight lines $ABC$, $ABD$ have the segment $AB$ common to both of them.

From the point $B$ draw $BE$ at right angles to $AB$.

Then, because $ABC$ is a straight line, [*Hypothesis.*
the angle $CBE$ is equal to the angle $EBA$. [*Definition* 10.

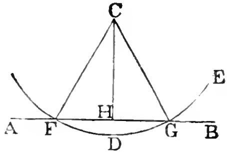
Also, because $ABD$ is a straight line, [*Hypothesis.*
the angle $DBE$ is equal to the angle $EBA$.
Therefore the angle $DBE$ is equal to the angle $CBE$, [*Ax.* 1.
the less to the greater; which is impossible. [*Axiom* 9.

Wherefore *two straight lines cannot have a common segment.*

PROPOSITION 12. *PROBLEM.*

*To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.*

Let $AB$ be the given straight line, which may be produced to any length both ways, and let $C$ be the given point without it: it is required to draw from the point $C$ a straight line perpendicular to $AB$.




Take any point $D$ on the other side of $AB$, and from the centre $C$, at the distance $CD$, describe the circle $EGF$, meeting $AB$ at $F$ and $G$. [*Postulate* 3.
Bisect $FG$ at $H$, [I. 10.
and join $CH$.

The straight line $CH$ drawn from the given point $C$ shall be perpendicular to the given straight line $AB$.

Join $CF$, $CG$.

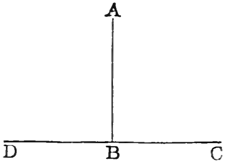
Because $FH$ is equal to $HG$, [*Construction.*
and $HC$ is common to the two triangles $FHC$, $GHC$;
the two sides $FH$, $HC$ are equal to the two sides $GH$, $HC$, each to each;
and the base $CF$ is equal to the base $CG$; [*Definition* 15.
therefore the angle $CHF$ is equal to the angle $CHG$; [I. 8.
and they are adjacent angles.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. [*Def.* 10.

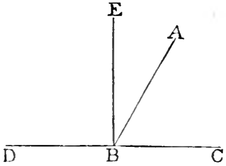
Wherefore *a perpendicular $CH$ has been drawn to the given straight line $AB$ from the given point $C$ without it.* Q.E.F.

## PROPOSITION 13. *THEOREM.*

*The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.*

Let the straight line *AB* make with the straight line *CD*, on one side of it, the angles *CBA*, *ABD*: these either are two right angles, or are together equal to two right angles.




For if the angle *CBA* is equal to the angle *ABD*, each of them is a right angle. [*Definition* 10.

But if not, from the point *B* draw *BE* at right angles to *CD*; [I. 11.

therefore the angles *CBE*, *EBD* are two right angles. [*Def.* 10.

Now the angle *CBE* is equal to the two angles *CBA*, *ABE*; to each of these equals add the angle *EBD*;

therefore the angles *CBE*, *EBD* are equal to the three angles *CBA*, *ABE*, *EBD*. [*Axiom* 2.

Again, the angle *DBA* is equal to the two angles *DBE*, *EBA*;

to each of these equals add the angle *ABC*;

therefore the angles *DBA*, *ABC* are equal to the three angles *DBE*, *EBA*, *ABC*. [*Axiom* 2.

But the angles *CBE*, *EBD* have been shewn to be equal to the same three angles.

Therefore the angles *CBE*, *EBD* are equal to the angles *DBA*, *ABC*. [*Axiom* 1

But *CBE*, *EBD* are two right angles;

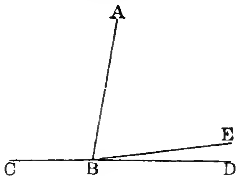
therefore *DBA*, *ABC* are together equal to two right angles.

Wherefore, *the angles* &c. Q.E.D.

PROPOSITION 14. *THEOREM.*

*If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.*

At the point $B$ in the straight line $AB$, let the two straight lines $BC$, $BD$, on the opposite sides of $AB$, make the adjacent angles $ABC$, $ABD$ together equal to two right angles: $BD$ shall be in the same straight line with $CB$.




For if $BD$ be not in the same straight line with $CB$, let $BE$ be in the same straight line with it.

Then because the straight line $AB$ makes with the straight line $CBE$, on one side of it, the angles $ABC$, $ABE$, these angles are together equal to two right angles. [I. 13.

But the angles $ABC$, $ABD$ are also together equal to two right angles. [*Hypothesis.*

Therefore the angles $ABC$, $ABE$ are equal to the angles $ABC$, $ABD$.

From each of these equals take away the common angle $ABC$, and the remaining angle $ABE$ is equal to the remaining angle $ABD$, [*Axiom* 3.

the less to the greater; which is impossible.

Therefore $BE$ is not in the same straight line with $CB$.

And in the same manner it may be shewn that no other can be in the same straight line with it but $BD$;

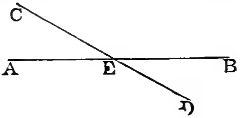
therefore $BD$ is in the same straight line with $CB$.

Wherefore, *if at a point* &c. Q.E.D.

PROPOSITION 15. *THEOREM.*

*If two straight lines cut one another, the vertical, or opposite, angles shall be equal.*

Let the two straight lines $AB$, $CD$ cut one another at the point $E$; the angle $AEC$ shall be equal to the angle $DEB$, and the angle $CEB$ to the angle $AED$.




Because the straight line $AE$ makes with the straight line $CD$ the angles $CEA$, $AED$, these angles are together equal to two right angles. [I. 13.
Again, because the straight line $DE$ makes with the straight line $AB$ the angles $AED$, $DEB$, these also are together equal to two right angles. [I. 13.
But the angles $CEA$, $AED$ have been shewn to be together equal to two right angles.
Therefore the angles $CEA$, $AED$ are equal to the angles $AED$, $DEB$.
From each of these equals take away the common angle $AED$, and the remaining angle $CEA$ is equal to the remaining angle $DEB$. [*Axiom* 3.

In the same manner it may be shewn that the angle $CEB$ is equal to the angle $AED$.

Wherefore, *if two straight lines* &c. Q.E.D.

Corollary 1. From this it is manifest that, if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

Corollary 2. And consequently, that all the angles made by any number of straight lines meeting at one point, are together equal to four right angles.

PROPOSITION 16. *THEOREM.*

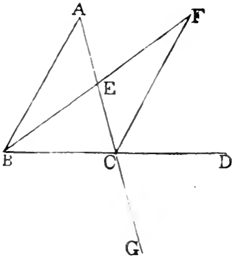
*If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.*

Let $ABC$ be a triangle, and let one side $BC$ be produced to $D$: the exterior angle $ACD$ shall be greater than either of the interior opposite angles $CBA$, $BAC$.

Bisect $AC$ at $E$, [I. 10.
join $BE$ and produce it to $F$, making $EF$ equal to $EB$, [I. 3.
and join $FC$.

Because $AE$ is equal to $EC$, and $BE$ to $EF$; [*Constr.*
the two sides $AE$, $EB$ are equal to the two sides $CE$, $EF$ each to each;

and the angle $AEB$ is equal to the angle $CEF$, because they are opposite vertical angles; [I. 15.




therefore the triangle $AEB$ is equal to the triangle $CEF$, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle $BAE$ is equal to the angle $ECF$.

But the angle $ECD$ is greater than the angle $ECF$. [*Axiom* 9.

Therefore the angle $ACD$ is greater than the angle $BAE$.

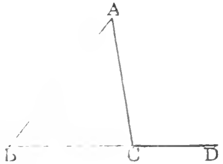
In the same manner if $BC$ be bisected, and the side $AC$ be produced to $G$, it may be shewn that the angle $BCG$, that is the angle $ACD$, is greater than the angle $ABC$. [I. 15.

Wherefore, *if one side* &c. Q.E.D.

PROPOSITION 17. *THEOREM.*

*Any two angles of a triangle are together less than two right angles.*

Let $ABC$ be a triangle: any two of its angles are together less than two right angles.




Produce $BC$ to $D$.

Then because $ACD$ is the exterior angle of the triangle $ABC$, it is greater than the interior opposite angle $ABC$. [I. 16.

To each of these add the angle $ACB$.

Therefore the angles $ACD$, $ACB$ are greater than the angles $ABC$, $ACB$.

But the angles $ACD$, $ACB$ are together equal to two right angles. [I. 13.

Therefore the angles $ABC$, $ACB$ are together less than two right angles.

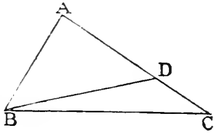
In the same manner it may be shewn that the angles $BAC$, $ACB$, as also the angles $CAB$, $ABC$, are together less than two right angles.

Wherefore, *any two angles* &c. Q.E.D.

## PROPOSITION 18. *THEOREM.*

*The greater side of every triangle has the greater angle opposite to it.*

Let $ABC$ be a triangle, of which the side $AC$ is greater than the side $AB$: the angle $ABC$ is also greater than the angle $ACB$.




Because $AC$ is greater than $AB$, make $AD$ equal to $AB$, [I. 3. and join $BD$.

Then, because $ADB$ is the exterior angle of the triangle $BDC$, it is greater than the interior opposite angle $DCB$. [I. 16.

But the angle $ADB$ is equal to the angle $ABD$, [I. 5. because the side $AD$ is equal to the side $AB$. [*Constr.*

Therefore the angle $ABD$ is also greater than the angle $ACB$.

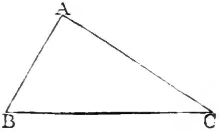
Much more then is the angle $ABC$ greater than the angle $ACB$. [*Axiom* 9.

Wherefore, *the greater side* &c. Q.E.D.

## PROPOSITION 19. *THEOREM.*

*The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.*

Let $ABC$ be a triangle, of which the angle $ABC$ is greater than the angle $ACB$: the side $AC$ is also greater than the side $AB$.




For if not, $AC$ must be either equal to $AB$ or less than $AB$.

But $AC$ is not equal to $AB$, for then the angle $ABC$ would be equal to the angle $ACB$; [I. 5. but it is not; [*Hypothesis.* therefore $AC$ is not equal to $AB$.

Neither is $AC$ less than $AB$, for then the angle $ABC$ would be less than the angle $ACB$; [I. 18. but it is not; [*Hypothesis.*

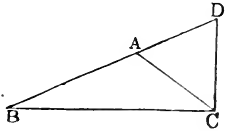
therefore $AC$ is not less than $AB$.
And it has been shewn that $AC$ is not equal to $AB$.
Therefore $AC$ is greater than $AB$.

Wherefore, *the greater angle* &c. Q.E.D.

PROPOSITION 20. *THEOREM.*

*Any two sides of a triangle are together greater than the third side.*

Let $ABC$ be a triangle: any two sides of it are together greater than the third side; namely, $BA$, $AC$ greater than $BC$; and $AB$, $BC$ greater than $AC$; and $BC$, $CA$ greater than $AB$.




Produce $BA$ to $D$, making $AD$ equal to $AC$, [I. 3.
and join $DC$.

Then, because $AD$ is equal to $AC$, [*Construction.*
the angle $ADC$ is equal to the angle $ACD$. [I. 5.
But the angle $BCD$ is greater than the angle $ACD$. [*Ax.* 9.
Therefore the angle $BCD$ is greater than the angle $BDC$.
And because the angle $BCD$ of the triangle $BCD$ is greater than its angle $BDC$, and that the greater angle is subtended by the greater side; [I. 19.
therefore the side $BD$ is greater than the side $BC$.
But $BD$ is equal to $BA$ and $AC$.
Therefore $BA$, $AC$ are greater than $BC$.

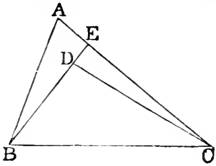
In the same manner it may be shewn that $AB$, $BC$ are greater than $AC$, and $BC$, $CA$ greater than $AB$.

Wherefore, *any two sides* &c. Q.E.D.

PROPOSITION 21. *THEOREM.*

*If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

Let $ABC$ be a triangle, and from the points $B$, $C$, the ends of the side $BC$, let the two straight lines $BD$, $CD$ be drawn to the point $D$ within the triangle: $BD$, $DC$ shall be less than the other two sides $BA$, $AC$ of the triangle, but shall contain an angle $BDC$ greater than the angle $BAC$.




Produce $BD$ to meet $AC$ at $E$.

Because two sides of a triangle are greater than the third side, the two sides $BA$, $AE$ of the triangle $ABE$ are greater than the side $BE$. [I. 20.

To each of these add $EC$.

Therefore $BA$, $AC$ are greater than $BE$, $EC$.

Again; the two sides $CE$, $ED$ of the triangle $CED$ are greater than the third side $CD$. [I. 20.

To each of these add $DB$.

Therefore $CE$, $EB$ are greater than $CD$, $DB$.

But it has been shewn that $BA$, $AC$ are greater than $BE$, $EC$;

much more then are $BA$, $AC$ greater than $BD$, $DC$.

Again, because the exterior angle of any triangle is greater than the interior opposite angle, the exterior angle $BDC$ of the triangle $CDE$ is greater than the angle $CED$. [I. 16.

For the same reason, the exterior angle $CEB$ of the triangle $ABE$ is greater than the angle $BAE$.

But it has been shewn that the angle $BDC$ is greater than the angle $CEB$;

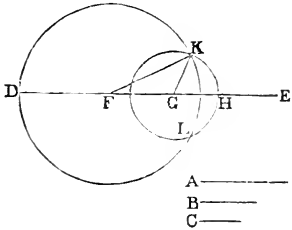
much more then is the angle $BDC$ greater than the angle $BAC$.

Wherefore, *if from the ends* &c. Q.E.D.

## PROPOSITION 22. *PROBLEM.*

***To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.***

Let $A$, $B$, $C$ be the three given straight lines, of which any two whatever are greater than the third; namely, $A$ and $B$ greater than $C$; $A$ and $C$ greater than $B$; and $B$ and $C$ greater than $A$: it is required to make a triangle of which the sides shall be equal to $A$, $B$, $C$, each to each.




Take a straight line $DE$ terminated at the point $D$, but unlimited towards $E$, and make $DF$ equal to $A$, $FG$ equal to $B$, and $GH$ equal to $C$. [I. 3.
From the centre $F$, at the distance $FD$, describe the circle $DKL$. [*Post.* 3.
From the centre $G$, at the distance $GH$, describe the circle $HLK$, cutting the former circle at $K$.
Join $KF$, $KG$. The triangle $KFG$ shall have its sides equal to the three straight lines $A$, $B$, $C$.

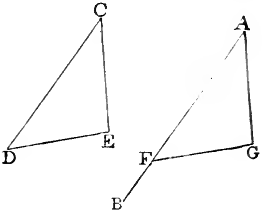
Because the point $F$ is the centre of the circle $DKL$, $FD$ is equal to $FK$. [*Definition* 15.
But $FD$ is equal to $A$. [*Construction.*
Therefore $FK$ is equal to $A$. [*Axiom* 1.
Again, because the point $G$ is the centre of the circle $HLK$, $GH$ is equal to $GK$. [*Definition* 15.
But $GH$ is equal to $C$. [*Construction.*
Therefore $GK$ is equal to $C$. [*Axiom* 1
And $FG$ is equal to $B$. [*Construction.*
Therefore the three straight lines $KF$, $FG$, $GK$ are equal to the three $A$, $B$, $C$.

Wherefore *the triangle* $KFG$ *has its three sides* $KF$, $FG$, $GK$ *equal to the three given straight lines* $A$, $B$, $C$. Q.E.F.

PROPOSITION 23. *PROBLEM.*

*At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.*

Let $AB$ be the given straight line, and $A$ the given point in it, and $DCE$ the given rectilineal angle: it is required to make at the given point $A$, in the given straight line $AB$, an angle equal to the given rectilineal angle $DCE$.




In $CD$, $CE$ take any points $D$, $E$, and join $DE$. Make the triangle $AFG$ the sides of which shall be equal to the three straight lines $CD$, $DE$, $EC$; so that $AF$ shall be equal to $CD$, $AG$ to $CE$, and $FG$ to $DE$. [I. 22.

The angle $FAG$ shall be equal to the angle $DCE$.

Because $FA$, $AG$ are equal to $DC$, $CE$, each to each, and the base $FG$ equal to the base $DE$; [*Construction.*

therefore the angle $FAG$ is equal to the angle $DCE$. [I. 8.

Wherefore *at the given point $A$ in the given straight line $AB$, the angle $FAG$ has been made equal to the given rectilineal angle $DCE$.* Q.E.F.

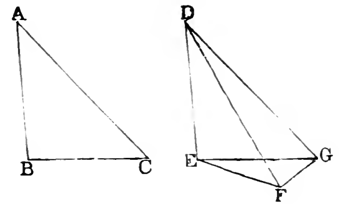
PROPOSITION 24. *THEOREM.*

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.*

Let $ABC$, $DEF$ be two triangles, which have the two sides $AB$, $AC$, equal to the two sides $DE$, $DF$, each to each, namely, $AB$ to $DE$, and $AC$ to $DF$, but the angle $BAC$ greater than the angle $EDF$: the base $BC$ shall be

greater than the base $EF$.

Of the two sides $DE$, $DF$, let $DE$ be the side which is not greater than the other. At the point $D$ in the straight line $DE$, make the angle $EDG$ equal to the angle $BAC$, [I. 23.




and make $DG$ equal to $AC$ or $DF$, [I. 3.
and join $EG$, $GF$.

Because $AB$ is equal to $DE$, [*Hypothesis.*
and $AC$ to $DG$; [*Construction.*
the two sides $BA$, $AC$ are equal to the two sides $ED$, $DG$, each to each;
and the angle $BAC$ is equal to the angle $EDG$; [*Constr.*
therefore the base $BC$ is equal to the base $EG$. [I. 4.

And because $DG$ is equal to $DF$, [*Construction.*
the angle $DGF$ is equal to the angle $DFG$. [I. 5.
But the angle $DGF$ is greater than the angle $EGF$. [*Ax* 9.
Therefore the angle $DFG$ is greater than the angle $EGF$.
Much more then is the angle $EFG$ greater than the angle $EGF$. [*Axiom* 9.

And because the angle $EFG$ of the triangle $EFG$ is greater than its angle $EGF$, and that the greater angle is subtended by the greater side, [I. 19.
therefore the side $EG$ is greater than the side $EF$.
But $EG$ was shewn to be equal to $BC$;
therefore $BC$ is greater than $EF$.

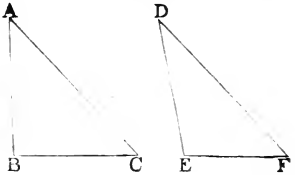
Wherefore, *if two triangles* &c. Q.E.D.

## PROPOSITION 25. *THEOREM.*

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one*

*greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.*

Let $ABC$, $DEF$ be two triangles, which have the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$, each to each, namely, $AB$ to $DE$, and $AC$ to $DF$, but the base $BC$ greater than the base $EF$: the angle $BAC$ shall be greater than the angle $EDF$.




For if not, the angle $BAC$ must be either equal to the angle $EDF$ or less than the angle $EDF$.

But the angle $BAC$ is not equal to the angle $EDF$, for then the base $BC$ would be equal to the base $EF$; [I. 4.
but it is not; [*Hypothesis.*
therefore the angle $BAC$ is not equal to the angle $EDF$.
Neither is the angle $BAC$ less than the angle $EDF$,
for then the base $BC$ would be less than the base $EF$; [I. 24.
but it is not; [*Hypothesis.*
therefore the angle $BAC$ is not less than the angle $EDF$.
And it has been shewn that the angle $BAC$ is not equal to the angle $EDF$.
Therefore the angle $BAC$ is greater than the angle $EDF$.

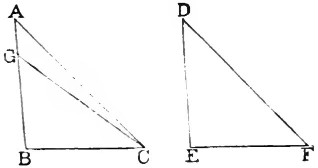
Wherefore, *if two triangles* &c. Q.E.D.

## PROPOSITION 26. *THEOREM.*

*If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.*

Let $ABC$, $DEF$ be two triangles, which have the angles $ABC$, $BCA$ equal to the angles $DEF$, $EFD$, each

to each, namely, $ABC$ to $DEF$, and $BCA$ to $EFD$; and let them have also one side equal to one side; and first let those sides be equal which are adjacent to the equal angles in the two triangles, namely, $BC$ to $EF$: the other sides shall be equal, each to each, namely, $AB$ to $DE$, and $AC$ to $DF$, and the third angle $BAC$ equal to the third angle $EDF$.




For if $AB$ be not equal to $DE$, one of them must be greater than the other. Let $AB$ be the greater, and make $BG$ equal to $DE$, [I. 3.

and join $GC$.

Then because $GB$ is equal to $DE$, [*Construction.*

and $BC$ to $EF$; [*Hypothesis.*

the two sides $GB$, $BC$ are equal to the two sides $DE$, $EF$, each to each;

and the angle $GBC$ is equal to the angle $DEF$; [*Hypothesis.*

therefore the triangle $GBC$ is equal to the triangle $DEF$, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle $GCB$ is equal to the angle $DFE$.

But the angle $DFE$ is equal to the angle $ACB$. [*Hypothesis.*

Therefore the angle $GCB$ is equal to the angle $ACB$, [*Ax.* 1.

the less to the greater; which is impossible.

Therefore $AB$ is not unequal to $DE$,

that is, it is equal to it;

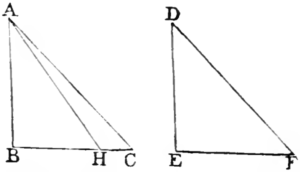
and $BC$ is equal to $EF$; [*Hypothesis.*

therefore the two sides $AB$, $BC$ are equal to the two sides $DE$, $EF$, each to each;

and the angle $ABC$ is equal to the angle $DEF$; [*Hypothesis.*

therefore the base $AC$ is equal to the base $DF$, and the third angle $BAC$ to the third angle $EDF$. [I. 4.

Next, let sides which are opposite to equal angles in each triangle be equal to one another, namely, $AB$ to $DE$: likewise in this case the other sides shall be equal, each to each, namely, $BC$ to $EF$, and $AC$ to $DF$, and also the third angle $BAC$ equal to the third angle $EDF$.




For if $BC$ be not equal to $EF$, one of them must be greater than the other.
Let $BC$ be the greater, and make $BH$ equal to $EF$, [I. 3.
and join $AH$.

Then because $BH$ is equal to $EF$, [*Construction.*
and $AB$ to $DE$; [*Hypothesis.*
the two sides $AB$, $BH$ are equal to the two sides $DE$, $EF$, each to each;
and the angle $ABH$ is equal to the angle $DEF$; [*Hypothesis.*
therefore the triangle $ABH$ is equal to the triangle $DEF$, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.
therefore the angle $BHA$ is equal to the angle $EFD$.
But the angle $EFD$ is equal to the angle $BCA$. [*Hypothesis.*
Therefore the angle $BHA$ is equal to the angle $BCA$; [*Ax.*1.
that is, the exterior angle $BHA$ of the triangle $AHC$ is equal to its interior opposite angle $BCA$;
which is impossible. [I. 16.
Therefore $BC$ is not unequal to $EF$,
that is, it is equal to it;
and $AB$ is equal to $DE$; [*Hypothesis.*
therefore the two sides $AB$, $BC$ are equal to the two sides $DE$, $EF$, each to each;
and the angle $ABC$ is equal to the angle $DEF$; [*Hypothesis.*
therefore the base $AC$ is equal to the base $DF$, and the third angle $BAC$ to the third angle $EDF$. [I. 4.

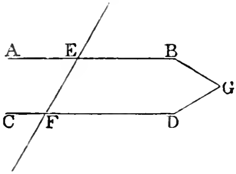
Wherefore, *if two triangles* &c. Q.E.D.

## PROPOSITION 27. *THEOREM.*

*If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.*

Let the straight line $EF$, which falls on the two straight lines $AB$, $CD$, make the alternate angles $AEF$, $EFD$ equal to one another: $AB$ shall be parallel to $CD$.

For if not, $AB$ and $CD$, being produced, will meet either towards $B$, $D$ or towards $A$, $C$. Let them be produced and meet towards $B$, $D$ at the point $G$.




Therefore $GEF$ is a triangle, and its exterior angle $AEF$ is greater than the interior opposite angle $EFG$; [I. 16.

But the angle $AEF$ is also equal to the angle $EFG$; [*Hyp.* which is impossible.

Therefore $AB$ and $CD$ being produced, do not meet towards $B$, $D$.

In the same manner, it may be shewn that they do not meet towards $A$, $C$.

But those straight lines which being produced ever so far both ways do not meet, are parallel. [*Definition* 35.

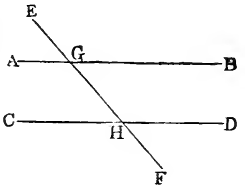
Therefore $AB$ is parallel to $CD$.

Wherefore, *if a straight line* &c. Q.E.D.

## PROPOSITION 28. *THEOREM.*

*If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.*

Let the straight line $EF$, which falls on the two straight lines $AB$, $CD$, make the exterior angle $EGB$ equal to the interior and opposite angle $GHD$ on the same side, or make the interior angles on the same side $BGH$, $GHD$ together equal to two right angles: $AB$ shall be parallel to $CD$.




Because the angle $EGB$ is equal to the angle $GHD$, [*Hyp.*
and the angle $EGB$ is also equal to the angle $AGH$, [I. 15.
therefore the angle $AGH$ is equal to the angle $GHD$; [*Ax.* 1.
and they are alternate angles;
therefore $AB$ is parallel to $CD$. [I. 27.

Again; because the angles $BGH$, $GHD$ are together equal to two right angles, [*Hypothesis.*
and the angles $AGH$, $BGH$ are also together equal to two right angles, [I. 13.
therefore the angles $AGH$, $BGH$ are equal to the angles $BGH$, $GHD$.
Take away the common angle $BGH$; therefore the remaining angle $AGH$ is equal to the remaining angle $GHD$; [*Axiom* 3.
and they are alternate angles;
therefore $AB$ is parallel to $CD$. [I. 27.

Wherefore, *if a straight line* &c. Q.E.D.

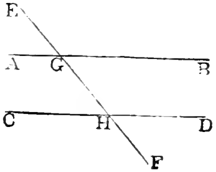
## PROPOSITION 29. *THEOREM.*

*If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.*

Let the straight line $EF$ fall on the two parallel straight lines $AB$, $CD$: the alternate angles $AGH$, $GHD$ shall be equal to one another, and the exterior angle $EGB$ shall be equal to the interior and opposite angle

on the same side, *GHD*, and the two interior angles on the same side, *BGH*, *GHD*, shall be together equal to two right angles.

For if the angle *AGH* be not equal to the angle *GHD*, one of them must be greater than the other; let the angle *AGH* be the greater.




Then the angle *AGH* is greater than the angle *GHD*;

to each of them add the angle *BGH*;

therefore the angles *AGH*, *BGH* are greater than the angles *BGH*, *GHD*.

But the angles *AGH*, *BGH* are together equal to two right angles; [I. 13.

therefore the angles *BGH*, *GHD* are together less than two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles. [*Axiom* 12.

Therefore the straight lines *AB*, *CD*, if continually produced, will meet.

But they never meet, since they are parallel by hypothesis.

Therefore the angle *AGH* is not unequal to the angle *GHD*; that is, it is equal to it.

But the angle *AGH* is equal to the angle *EGB*. [I. 15.

Therefore the angle *EGB* is equal to the angle *GHD*. [*Ax.* 1

Add to each of these the angle *BGH*.

Therefore the angles *EGB*, *BGH* are equal to the angles *BGH*, *GHD*. [*Axiom* 2.

But the angles *EGB*, *BGH* are together equal to two right angles. [I. 13

Therefore the angles *BGH*, *GHD* are together equal to two right angles. [*Axiom* 1.

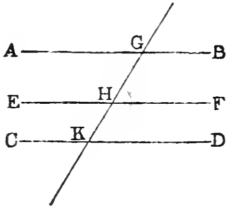
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 30. *THEOREM.*

*Straight lines which are parallel to the same straight line are parallel to each other.*

Let $AB$, $CD$ be each of them parallel to $EF$: $AB$ shall be parallel to $CD$.

Let the straight line $GHK$ cut $AB$, $EF$, $CD$.




Then, because $GHK$ cuts the parallel straight lines $AB$, $EF$, the angle $AGH$ is equal to the angle $GHF$. [I. 29.

Again, because $GK$ cuts the parallel straight lines $EF$, $CD$, the angle $GHF$ is equal to the angle $GKD$. [I. 29.

And it was shewn that the angle $AGK$ is equal to the angle $GHF$.

Therefore the angle $AGK$ is equal to the angle $GKD$; [*Ax.* 1.

and they are alternate angles;

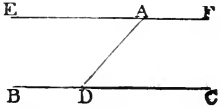
therefore $AB$ is parallel to $CD$. [I. 27.

Wherefore, *straight lines* &c. Q.E.D.

PROPOSITION 31. *PROBLEM.*

*To draw a straight line through a given point parallel to a given straight line.*

Let $A$ be the given point, and $BC$ the given straight line: it is required to draw a straight line through the point $A$ parallel to the straight line $BC$.




In $BC$ take any point $D$, and join $AD$; at the point $A$ in the straight line $AD$, make the angle $DAE$ equal to the angle $ADC$; [I. 23.

and produce the straight line $EA$ to $F$.

$EF$ shall be parallel to $BC$.

Because the straight line $AD$, which meets the two straight lines $BC$, $EF$, makes the alternate angles $EAD$, $ADC$ equal to one another, [*Construction.*

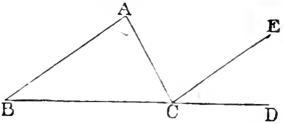
$EF$ is parallel to $BC$. [I. 27.

Wherefore *the straight line $EAF$ is drawn through the given point $A$, parallel to the given straight line $BC$.* Q.E.F.

PROPOSITION 32. *THEOREM.*

*If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.*

Let $ABC$ be a triangle, and let one of its sides $BC$ be produced to $D$: the exterior angle $ACD$ shall be equal to the two interior and opposite angles $CAB$, $ABC$; and the three interior angles of the triangle, namely, $ABC$, $BCA$, $CAB$ shall be equal to two right angles.




Through the point $C$ draw $CE$ parallel to $AB$. [I. 31.

Then, because $AB$ is parallel to $CE$, and $AC$ falls on them, the alternate angles $BAC$, $ACE$ are equal. [I. 29.

Again, because $AB$ is parallel to $CE$, and $BD$ falls on them, the exterior angle $ECD$ is equal to the interior and opposite angle $ABC$. [I. 29.

But the angle $ACE$ was shewn to be equal to the angle $BAC$;

therefore the whole exterior angle $ACD$ is equal to the two interior and opposite angles $CAB$, $ABC$. [*Axiom* 2.

To each of these equals add the angle $ACB$;

therefore the angles $ACD$, $ACB$ are equal to the three angles $CBA$, $BAC$, $ACB$. [*Axiom* 2.

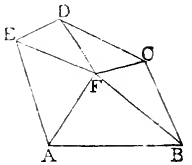
But the angles $ACD$, $ACB$ are together equal to two right angles; [I. 13.

therefore also the angles $CBA$, $BAC$, $ACB$ are together equal to two right angles. [*Axiom* 1.

Wherefore, *if a side of any triangle* &c. Q.E.D.

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure $ABCDE$ can be divided into as many triangles as the figure has sides, by drawing straight lines from a point $F$ within the figure to each of its angles.

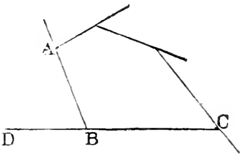



And by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as the figure has sides.

And the same angles are equal to the interior angles of the figure, together with the angles at the point $F$, which is the common vertex of the triangles, that is, together with four right angles. [I. 15. *Corollary* 2.

Therefore all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

COROLLARY 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*




Because every interior angle $ABC$, with its adjacent exterior angle $ABD$, is equal to two right angles; [I. 13.

therefore all the interior angles of the figure, together with all its exterior angles, are equal to twice as many right angles as the figure has sides.

But, by the foregoing Corollary all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Therefore all the interior angles of the figure, together with all its exterior angles, are equal to all the interior angles of the figure, together with four right angles.

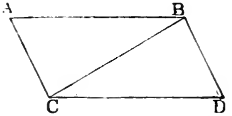
Therefore all the exterior angles are equal to four right angles.

PROPOSITION 33. *THEOREM.*

*The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.*

Let $AB$ and $CD$ be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines $AC$ and $BD$: $AC$ and $BD$ shall be equal and parallel.

Join $BC$.




Then because $AB$ is parallel to $CD$, [*Hypothesis.*
and $BC$ meets them,
the alternate angles $ABC$, $BCD$ are equal. [I. 29.
And because $AB$ is equal to $CD$, [*Hypothesis.*
and $BC$ is common to the two triangles $ABC$, $DCB$; the two sides $AB$, $BC$ are equal to the two sides $DC$, $CB$, each to each;
and the angle $ABC$ was shewn to be equal to the angle $BCD$;
therefore the base $AC$ is equal to the base $BD$, and the triangle $ABC$ to the triangle $BCD$, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.
therefore the angle $ACB$ is equal to the angle $CBD$.

And because the straight line $BC$ meets the two straight lines $AC$, $BD$, and makes the alternate angles $ACB$, $CBD$ equal to one another, $AC$ is parallel to $BD$. [I. 27.
And it was shewn to be equal to it.

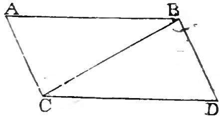
Wherefore, *the straight lines* &c. Q.E.D.

PROPOSITION 34. *THEOREM.*

*The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram, that is, divides it into two equal parts.*

*Note.* A parallelogram is a four-sided figure of which the opposite sides are parallel; and a diameter is the straight line joining two of its opposite angles.

Let $ACDB$ be a parallelogram, of which $BC$ is a diameter; the opposite sides and angles of the figure shall be equal to one another, and the diameter $BC$ shall bisect it.




Because $AB$ is parallel to $CD$, and $BC$ meets them, the alternate angles $ABC$, $BCD$ are equal to one another. [I. 29.

And because $AC$ is parallel to $BD$, and $BC$ meets them, the alternate angles $ACB$, $CBD$ are equal to one another. [I. 29.

Therefore the two triangles $ABC$, $BCD$ have two angles $ABC$, $BCA$ in the one, equal to two angles $DCB$, $CBD$ in the other, each to each, and one side $BC$ is common to the two triangles, which is adjacent to their equal angles;

therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, namely, the side $AB$ equal to the side $CD$, and the side $AC$ equal to the side $BD$, and the angle $BAC$ equal to the angle $CDB$. [I. 26.

And because the angle $ABC$ is equal to the angle $BCD$, and the angle $CBD$ to the angle $ACB$,

the whole angle $ABD$ is equal to the whole angle $ACD$. [*Ax.* 2.

And the angle $BAC$ has been shewn to be equal to the angle $CDB$.

Therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter bisects the parallelogram.

For $AB$ being equal to $CD$, and $BC$ common,

the two sides $AB$, $BC$ are equal to the two sides $DC$, $CB$ each to each;

and the angle $ABC$ has been shewn to be equal to the angle $BCD$;

therefore the triangle $ABC$ is equal to the triangle $BCD$, [I. 4.

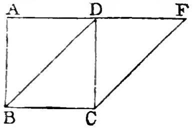
and the diameter $BC$ divides the parallelogram $ACDB$ into two equal parts.

Wherefore, *the opposite sides* &c. Q.E.D.

PROPOSITION 35. *THEOREM.*

*Parallelograms on the same base, and between the same parallels, are equal to one another.*

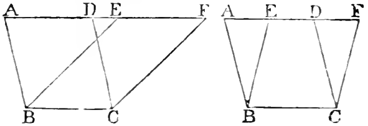
Let the parallelograms $ABCD$, $EBCF$ be on the same base $BC$, and between the same parallels $AF$, $BC$: the parallelogram $ABCD$ shall be equal to the parallelogram $EBCF$.




If the sides $AD$, $DF$ of the parallelograms $ABCD$, $DBCF$, opposite to the base $BC$, be terminated at the same point $D$, it is plain that each of the parallelograms is double of the triangle $BDC$; [I. 34.

and they are therefore equal to one another. [*Axiom* 6.

But if the sides $AD$, $EF$, opposite to the base $BC$ of the parallelograms $ABCD$, $EBCF$ be not terminated at the same point, then, because $ABCD$ is a parallelogram $AD$ is equal to $BC$; [I. 34.




for the same reason $EF$ is equal to $BC$;

therefore $AD$ is equal to $EF$; [*Axiom* 1.

therefore the whole, or the remainder, $AE$ is equal to the whole, or the remainder, $DF$. [*Axioms* 2, 3.

And $AB$ is equal to $DC$; [I. 34.

therefore the two sides $EA$, $AB$ are equal to the two sides $FD$, $DC$ each to each;

and the exterior angle $FDC$ is equal to the interior and opposite angle $EAB$; [I. 29.

therefore the triangle $EAB$ is equal to the triangle $FDC$. [I. 4.

Take the triangle $FDC$ from the trapezium $ABCF$, and from the same trapezium take the triangle $EAB$, and the remainders are equal; [*Axiom* 3.

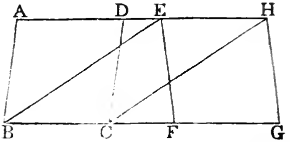
that is, the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

Wherefore, *parallelograms on the same base* &c. Q.E.D.

PROPOSITION 36. *THEOREM.*

*Parallelograms on equal bases, and between the same parallels, are equal to one another.*

Let $ABCD$, $EFGH$ be parallelograms on equal bases $BC$, $FG$, and between the same parallels $AH$, $BG$: the parallelogram $ABCD$ shall be equal to the parallelogram $EFGH$.




Join $BE$, $CH$.

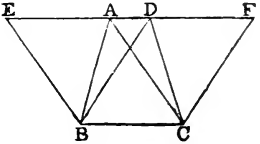
Then, because $BC$ is equal to $FG$, [*Hyp.*
and $FG$ to $EH$, [I. 34.
$BC$ is equal to $EH$; [*Axiom* 1.
and they are parallels, [*Hypothesis.*
and joined towards the same parts by the straight lines $BE$, $CH$.
But straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel. [I. 33.
Therefore $BE$, $CH$ are both equal and parallel.
Therefore $EBCH$ is a parallelogram. [*Definition.*
And it is equal to $ABCD$, because they are on the same base $BC$, and between the same parallels $BC$, $AH$. [I. 35.

For the same reason the parallelogram $EFGH$ is equal to the same $EBCH$.
Therefore the parallelogram $ABCD$ is equal to the parallelogram $EFGH$. [*Axiom* 1.

Wherefore, *parallelograms* &c. Q.E.D.

PROPOSITION 37. *THEOREM.*

*Triangles on the same base, and between the same parallels, are equal.*




Let the triangles $ABC$, $DBC$ be on the same base $BC$, and between the same parallels $AD$, $BC$: the triangle $ABC$ shall be equal to the triangle $DBC$.

Produce $AD$ both ways to the points $E$, $F$; [*Post.* 2.

through $B$ draw $BE$ parallel to $CA$, and through $C$ draw $CF$ parallel to $BD$. [I. 31.

Then each of the figures $EBCA$, $DBCF$ is a parallelogram; [*Definition.*

and $EBCA$ is equal to $DBCF$, because they are on the same base $BC$, and between the same parallels $BC$, $EF$. [I. 35.

And the triangle $ABC$ is half of the parallelogram $EBCA$, because the diameter $AB$ bisects the parallelogram; [I. 34.

and the triangle $DBC$ is half of the parallelogram $DBCF$, because the diameter $DC$ bisects the parallelogram. [I. 34.

But the halves of equal things are equal. [*Axiom* 7.

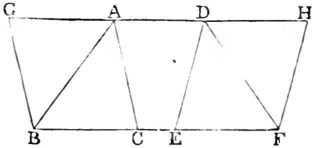
Therefore the triangle $ABC$ is equal to the triangle $DBC$.

Wherefore, *triangles* &c. Q.E.D.

PROPOSITION 38. *THEOREM.*

*Triangles on equal bases, and between the same parallels, are equal to one another.*

Let the triangles $ABC$, $DEF$ be on equal bases $BC$, $EF$, and between the same parallels $BF$, $AD$: the triangle $ABC$ shall be equal to the triangle $DEF$.




Produce $AD$ both ways to the points $G$, $H$;

through $B$ draw $BG$ parallel to $CA$, and through $F$ draw $FH$ parallel to $ED$. [I. 31.

Then each of the figures $GBCA$, $DEFH$ is a parallelogram. [*Definition.*

And they are equal to one another because they are on equal bases $BC$, $EF$, and between the same parallels $BF$, $GH$. [I. 36.

And the triangle $ABC$ is half of the parallelogram $GBCA$, because the diameter $AB$ bisects the parallelogram; [I. 34.

and the triangle $DEF$ is half of the parallelogram $DEFH$, because the diameter $DF$ bisects the parallelogram.

But the halves of equal things are equal. [*Axiom* 7.

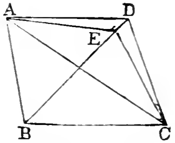
Therefore the triangle $ABC$ is equal to the triangle $DEF$.

Wherefore, *triangles* &c. Q.E.D.

PROPOSITION 39. *THEOREM.*

*Equal triangles on the same base, and on the same side of it, are between the same parallels.*

Let the equal triangles $ABC$, $DBC$ be on the same base $BC$, and on the same side of it: they shall be between the same parallels.



Join $AD$.

$AD$ shall be parallel to $BC$.

For if it is not, through $A$ draw $AE$ parallel to $BC$, meeting $BD$ at $E$. [I. 31.

and join $EC$.

Then the triangle $ABC$ is equal to the triangle $EBC$, because they are on the same base $BC$, and between the same parallels $BC$, $AE$. [I. 37.

But the triangle $ABC$ is equal to the triangle $DBC$. [*Hyp.*

Therefore also the triangle $DBC$ is equal to the triangle $EBC$, [*Axiom* 1.

the greater to the less; which is impossible.

Therefore $AE$ is not parallel to $BC$.

In the same manner it can be shewn that no other straight line through $A$ but $AD$ is parallel to $BC$;

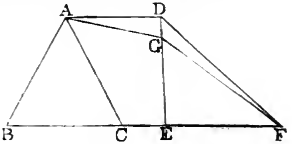
therefore $AD$ is parallel to $BC$.

Wherefore, *equal triangles* &c. Q.E.D.

PROPOSITION 40. *THEOREM.*

*Equal triangles, on equal bases, in the same straight line, and on the same side of it, are between the same parallels.*

Let the equal triangles $ABC$, $DEF$ be on equal bases $BC$, $EF$, in the same straight line $BF$, and on the same side of it: they shall be between the same parallels.



Join $AD$.

$AD$ shall be parallel to $BF$.

For if it is not, through $A$ draw $AG$ parallel to $BF$, meeting $ED$ at $G$ [I. 31.

and join $GF$.

Then the triangle $ABC$ is equal to the triangle $GEF$, because they are on equal bases $BC$, $EF$, and between the same parallels. [I. 38.

But the triangle $ABC$ is equal to the triangle $DEF$. [*Hyp.*

Therefore also the triangle $DEF$ is equal to the triangle $GEF$, [*Axiom* 1.

the greater to the less; which is impossible.

Therefore $AG$ is not parallel to $BF$.

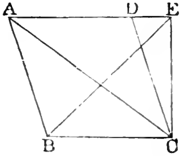
In the same manner it can be shewn that no other straight line through $A$ but $AD$ is parallel to $BF$; therefore $AD$ is parallel to $BF$.

Wherefore, *equal triangles* &c. Q.E.D.

## PROPOSITION 41. *THEOREM.*

*If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.*

Let the parallelogram $ABCD$ and the triangle $EBC$ be on the same base $BC$, and between the same parallels $BC$, $AE$: the parallelogram $ABCD$ shall be double of the triangle $EBC$.




Join $AC$.

Then the triangle $ABC$ is equal to the triangle $EBC$, because they are on the same base $BC$, and between the same parallels $BC$, $AE$. [I. 37.

But the parallelogram $ABCD$ is double of the triangle $ABC$, because the diameter $AC$ bisects the parallelogram. [I. 34.

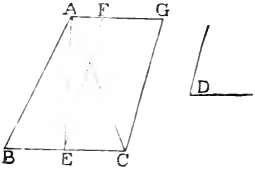
Therefore the parallelogram $ABCD$ is also double of the triangle $EBC$.

Wherefore, *if a parallelogram* &c. Q.E.D.

PROPOSITION 42. *PROBLEM.*

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let $ABC$ be the given triangle, and $D$ the given rectilineal angle: it is required to describe a parallelogram that shall be equal to the given triangle $ABC$, and have one of its angles equal to $D$.




Bisect $BC$ at $E$: [I. 10.
join $AE$, and at the point $E$, in the straight line $EC$, make the angle $CEF$ equal to $D$; [I. 23.
through $A$ draw $AFG$ parallel to $EC$, and through $C$ draw $CG$ parallel to $EF$. [I. 31.

Therefore $FECG$ is a parallelogram. [*Definition.*

And, because $BE$ is equal to $EC$, [*Construction.*
the triangle $ABE$ is equal to the triangle $AEC$, because they are on equal bases $BE$, $EC$, and between the same parallels $BC$, $AG$. [I. 38.

Therefore the triangle $ABC$ is double of the triangle $AEC$.

But the parallelogram $FECG$ is also double of the triangle $AEC$, because they are on the same base $EC$, and between the same parallels $EC$, $AG$. [I. 41.

Therefore the parallelogram $FECG$ is equal to the triangle $ABC$; [*Axiom* 6.

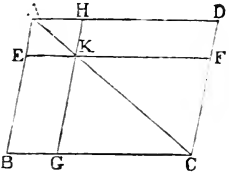
and it has one of its angles $CEF$ equal to the given angle $D$. [*Construction.*

Wherefore *a parallelogram FECG has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D.* Q.E.F.

## PROPOSITION 43. *THEOREM.*

*The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.*

Let $ABCD$ be a parallelogram, of which the diameter is $AC$; and $EH$, $GF$ parallelograms about $AC$, that is, through which $AC$ passes; and $BK$, $KD$ the other parallelograms which make up the whole figure $ABCD$, and which are therefore called the complements: the complement $BK$ shall be equal to the complement $KD$.




Because $ABCD$ is a parallelogram, and $AC$ its diameter, the triangle $ABC$ is equal to the triangle $ADC$. [I. 34.

Again, because $AEKH$ is a parallelogram, and $AK$ its diameter, the triangle $AEK$ is equal to the triangle $AHK$. [I. 34.

For the same reason the triangle $KGC$ is equal to the triangle $KFC$.

Therefore, because the triangle $AEK$ is equal to the triangle $AHK$, and the triangle $KGC$ to the triangle $KFC$; the triangle $AEK$ together with the triangle $KGC$ is equal to the triangle $AHK$ together with the triangle $KFC$. [*Ax.* 2.

But the whole triangle $ABC$ was shewn to be equal to the whole triangle $ADC$.

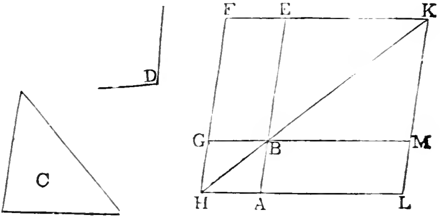
Therefore the remainder, the complement $BK$, is equal to the remainder, the complement $KD$. [*Axiom* 3.

Wherefore, *the complements* &c. Q.E.D.

## PROPOSITION 44. *PROBLEM.*

*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let $AB$ be the given straight line, and $C$ the given triangle, and $D$ the given rectilineal angle: it is required to apply to the straight line $AB$ a parallelogram equal to the triangle $C$, and having an angle equal to $D$.




Make the parallelogram $BEFG$ equal to the triangle $C$, and having the angle $EBG$ equal to the angle $D$, so that $BE$ may be in the same straight line with $AB$; [I. 42.
produce $FG$ to $H$;
through $A$ draw $AH$ parallel to $BG$ or $EF$, [I. 31.
and join $HB$.

Then, because the straight line $HF$ falls on the parallels $AH$, $EF$, the angles $AHF$, $HFE$ are together equal to two right angles. [I. 29.
Therefore the angles $BHF$, $HFE$ are together less than two right angles.
But straight lines which with another straight line make the interior angles on the same side together less than two right angles will meet on that side, if produced far enough. [*Ax.* 12.
Therefore $HB$ and $FE$ will meet if produced;
let them meet at $K$.
Through $K$ draw $KL$ parallel to $EA$ or $FH$; [I. 31.
and produce $HA$, $GB$ to the points $L$, $M$.

Then $HLKF$ is a parallelogram, of which the diameter is $HK$; and $AG$, $ME$ are parallelograms about $HK$; and $LB$, $BF$ are the complements.
Therefore $LB$ is equal to $BF$. [I. 43.
But $BF$ is equal to the triangle $C$. [*Construction.*
Therefore $LB$ is equal to the triangle $C$. [*Axiom* 1.

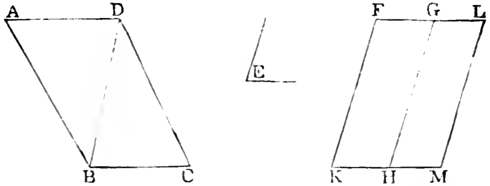
And because the angle $GBE$ is equal to the angle $ABM$, [I.15.
and likewise to the angle $D$; [*Construction.*
the angle $ABM$ is equal to the angle $D$. [*Axiom* 1.

Wherefore *to the given straight line $AB$ the parallelogram $LB$ is applied, equal to the triangle $C$, and having the angle $ABM$ equal to the angle $D$.* Q.E.F.

PROPOSITION 45. *PROBLEM.*

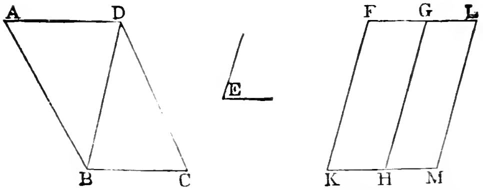
*To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.*

Let $ABCD$ be the given rectilineal figure, and $E$ the given rectilineal angle: it is required to describe a parallelogram equal to $ABCD$, and having an angle equal to $E$.




Join $DB$, and describe the parallelogram $FH$ equal to the triangle $ADB$, and having the angle $FKH$ equal to the angle $E$; [I. 42.
and to the straight line $GH$ apply the parallelogram $GM$ equal to the triangle $DBC$, and having the angle $GHM$ equal to the angle $E$. [I. 44.
The figure $FKML$ shall be the parallelogram required.

Because the angle $E$ is equal to each of the angles $FKH$, $GHM$, [*Construction.*
the angle $FKH$ is equal to the angle $GHM$. [*Axiom* 1.
Add to each of these equals the angle $KHG$;
therefore the angles $FKH$, $KHG$ are equal to the angles $KHG$, $GHM$. [*Axiom* 2.
But $FKH$, $KHG$ are together equal to two right angles; [I.29.
therefore $KHG$, $GHM$ are together equal to two right angles.




And because at the point $H$ in the straight line $GH$, the two straight lines $KH$, $HM$, on the opposite sides of it, make the adjacent angles together equal to two right angles, $KH$ is in the same straight line with $HM$. [I. 14.

And because the straight line $HG$ meets the parallels $KM$, $FG$, the alternate angles $MHG$, $HGF$ are equal. [I. 29.

Add to each of these equals the angle $HGL$;

therefore the angles $MHG$, $HGL$, are equal to the angles $HGF$, $HGL$. [*Axiom* 2.

But $MHG$, $HGL$ are together equal to two right angles; [I. 29.

therefore $HGF$, $HGL$ are together equal to two right angles

Therefore $FG$ is in the same straight line with $GL$. [I. 14.

And because $KF$ is parallel to $HG$, and $HG$ to $ML$, [*Constr.*

$KF$ is parallel to $ML$; [I. 30.

and $KM$, $FL$ are parallels; [*Construction.*

therefore $KFLM$ is a parallelogram. [*Definition.*

And because the triangle $ABD$ is equal to the parallelogram $HF$, [*Construction.*

and the triangle $DBC$ to the parallelogram $GM$; [*Constr.*

the whole rectilineal figure $ABCD$ is equal to the whole parallelogram $KFLM$. [*Axiom* 2

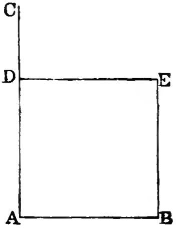
Wherefore, *the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E.* Q.E.F.

Corollary. From this it is manifest, how to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; namely, by applying to the given straight line a parallelogram equal to the first triangle $ABD$, and having an angle equal to the given angle; and so on. [I. 44.

## PROPOSITION 46. *PROBLEM.*

*To describe a square on a given straight line.*

Let $AB$ be the given straight line: it is required to describe a square on $AB$.



From the point $A$ draw $AC$ at right angles to $AB$; [I. 11.
and make $AD$ equal to $AB$; [I. 3.
through $D$ draw $DE$ parallel to $AB$; and through $B$ draw $BE$ parallel to $AD$. [I. 31.
$ADEB$ shall be a square.

For $ADEB$ is by construction a parallelogram;
therefore $AB$ is equal to $DE$, and $AD$ to $BE$. [I. 34.
But $AB$ is equal to $AD$. [*Construction.*
Therefore the four straight lines $BA$, $AD$, $DE$, $EB$ are equal to one another, and the parallelogram $ADEB$ is equilateral. [*Axiom* 1.

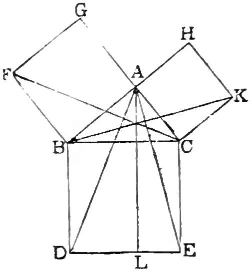
Likewise all its angles are right angles.
For since the straight line $AD$ meets the parallels $AB$, $DE$, the angles $BAD$, $ADE$ are together equal to two right angles; [I. 29.
but $BAD$ is a right angle; [*Construction.*
therefore also $ADE$ is a right angle. [*Axiom* 3.
But the opposite angles of parallelograms are equal. [I. 34.
Therefore each of the opposite angles $ABE$, $BED$ is a right angle. [*Axiom* 1.
Therefore the figure $ADEB$ is rectangular;
and it has been shewn to be equilateral.
Therefore *it is a square.* [*Definition* 30.
*And it is described on the given straight line $AB$.* Q.E.F.

COROLLARY. From the demonstration it is manifest that every parallelogram which has one right angle has all its angles right angles.

## PROPOSITION 47. *THEOREM.*

*In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.*

Let *ABC* be a right-angled triangle, having the right angle *BAC*: the square described on the side *BC* shall be equal to the squares described on the sides *BA*, *AC*.




On *BC* describe the square *BDEC*, and on *BA*, *AC* describe the squares *GB*, *HC*; [I. 46.
through *A* draw *AL* parallel to *BD* or *CE*; [I. 31.
and join *AD*, *FC*.

Then, because the angle *BAC* is a right angle, [*Hypothesis.*
and that the angle *BAG* is also a right angle, [*Definition* 30.
the two straight lines *AC*, *AG*, on the opposite sides of *AB*, make with it at the point *A* the adjacent angles equal to two right angles;
therefore *CA* is in the same straight line with *AG*. [I. 14.
For the same reason, *AB* and *AH* are in the same straight line.

Now the angle *DBC* is equal to the angle *FBA*, for each of them is a right angle. [*Axiom* 11.
Add to each the angle *ABC*.
Therefore the whole angle *DBA* is equal to the whole angle *FBC*. [*Axiom* 2.
And because the two sides *AB*, *BD* are equal to the two sides *FB*, *BC*, each to each; [*Definition* 30.
and the angle *DBA* is equal to the angle *FBC*;
therefore the triangle *ABD* is equal to the triangle *FBC*. [I. 4.

Now the parallelogram $BL$ is double of the triangle $ABD$, because they are on the same base $BD$, and between the same parallels $BD$, $AL$. [I. 41.

And the square $GB$ is double of the triangle $FBC$, because they are on the same base $FB$, and between the same parallels $FB$, $GC$. [I. 41.

But the doubles of equals are equal to one another. [*Ax.* 6.

Therefore the parallelogram $BL$ is equal to the square $GB$.

In the same manner, by joining $AE$, $BK$, it can be shewn, that the parallelogram $CL$ is equal to the square $CH$.

Therefore the whole square $BDEC$ is equal to the two squares $GB$, $HC$. [*Axiom* 2.

And the square $BDEC$ is described on $BC$, and the squares $GB$, $HC$ on $BA$, $AC$.

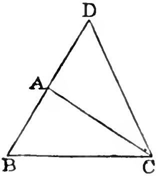
Therefore the square described on the side $BC$ is equal to the squares described on the sides $BA$, $AC$.

Wherefore, *in any right-angled triangle* &c. Q.E.D.

## PROPOSITION 48. *THEOREM.*

*If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.*

Let the square described on $BC$, one of the sides of the triangle $ABC$, be equal to the squares described on the other sides $BA$, $AC$: the angle $BAC$ shall be a right angle.




From the point $A$ draw $AD$ at right angles to $AC$; [I. 11.

and make $AD$ equal to $BA$; [I. 3.

and join $DC$.

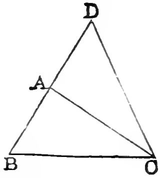
Then because $DA$ is equal to $BA$, the square on $DA$ is equal to the square on $BA$.

To each of these add the square on $AC$.

Therefore the squares on $DA$, $AC$ are equal to the squares on $BA$, $AC$. [*Axiom* 2.

But because the angle $DAC$ is a right angle, [*Construction.*
the square on $DC$ is equal to the squares on $DA$, $AC$. [I. 47.
And, by hypothesis, the square on $BC$ is equal to the squares on $BA$, $AC$.
Therefore the square on $DC$ is equal to the square on $BC$. [*Ax.* 1.
Therefore also the side $DC$ is equal to the side $BC$.

And because the side $DA$ is equal to the side $AB$; [*Constr.*
and the side $AC$ is common to the two triangles $DAC$, $BAC$;
the two sides $DA$, $AC$ are equal to the two sides $BA$, $AC$, each to each;
and the base $DC$ has been shewn to be equal to the base $BC$;




therefore the angle $DAC$ is equal to the angle $BAC$. [I. 8.
But $DAC$ is a right angle; [*Construction.*
therefore also $BAC$ is a right angle. [*Axiom* 1.

Wherefore, *if the square* &c. Q.E.D.

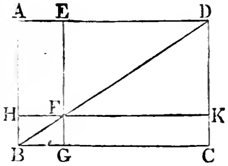
# *BOOK II.*

## DEFINITIONS.

1. EVERY right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.

Thus the parallelogram $HG$, together with the complements $AF$, $FC$, is the gnomon, which is more briefly expressed by the letters $AGK$, or $EHC$, which are at the opposite angles of the parallelograms which make the gnomon.

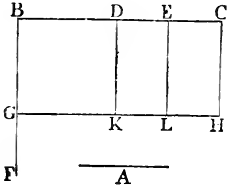



PROPOSITION 1. *THEOREM.*

*If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.*

Let $A$ and $BC$ be two straight lines; and let $BC$ be divided into any number of parts at the points $D$, $E$: the rectangle contained by the straight lines $A$, $BC$, shall be equal to the rectangle contained by $A$, $BD$, together with that contained by $A$, $DE$, and that contained by $A$, $EC$.

From the point $B$ draw $BF$ at right angles to $BC$; [I. 11. and make $BG$ equal to $A$; [I. 3. through $G$ draw $GH$ parallel to $BC$; and through $D$, $E$, $C$ draw $DK$, $EL$, $CH$, parallel to $BG$. [I. 31.




Then the rectangle $BH$ is equal to the rectangles $BK$, $DL$, $EH$.

But $BH$ is contained by $A$, $BC$, for it is contained by $GB$, $BC$, and $GB$ is equal to $A$. [*Construction.*

And $BK$ is contained by $A$, $BD$, for it is contained by $GB$, $BD$, and $GB$ is equal to $A$;

and $DL$ is contained by $A$, $DE$, because $DK$ is equal to $BG$, which is equal to $A$; [I. 34.

and in like manner $EH$ is contained by $A$, $EC$.

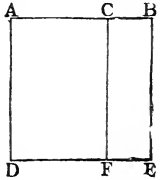
Therefore the rectangle contained by $A$, $BC$ is equal to the rectangles contained by $A$, $BD$, and by $A$, $DE$, and by $A$, $EC$.

**Wherefore,** *if there be two straight lines* &c. Q.E.D.

## PROPOSITION 2. *THEOREM.*

*If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square on the whole line.*

Let the straight line *AB* be divided into any two parts at the point *C*: the rectangle contained by *AB*, *BC*, together with the rectangle *AB*, *AC*, shall be equal to the square on *AB*.




[*Note.* To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines *AB*, *AC* is sometimes simply called the rectangle *AB*, *AC*.]

On *AB* describe the square *ADEB*; [I. 46.

and through *C* draw *CF* parallel to *AD* or *BE*. [I. 31.

Then *AE* is equal to the rectangles *AF*, *CE*.

But *AE* is the square on *AB*.

And *AF* is the rectangle contained by *BA*, *AC*, for it is contained by *DA*, *AC*, of which *DA* is equal to *BA*;

and *CE* is contained by *AB*, *BC*, for *BE* is equal to *AB*.

Therefore the rectangle *AB*, *AC*, together with the rectangle *AB*, *BC*, is equal to the square on *AB*.

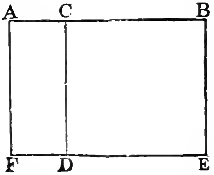
Wherefore, *if a straight line* &c. Q.E.D.

## PROPOSITION 3. *THEOREM.*

*If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.*

Let the straight line *AB* be divided into any two parts at the point *C*: the rectangle *AB*, *BC* shall be equal to the rectangle *AC*, *CB*, together with the square on *BC*.

On $BC$ describe the square $CDEB$; [I. 46.
produce $ED$ to $F$, and through $A$ draw $AF$ parallel to $CD$ or $BE$. [I. 31.




Then the rectangle $AE$ is equal to the rectangles $AD$, $CE$.
But $AE$ is the rectangle contained by $AB$, $BC$, for it is contained by $AB$, $BE$, of which $BE$ is equal to $BC$;
and $AD$ is contained by $AC$, $CB$, for $CD$ is equal to $CB$; and $CE$ is the square on $BC$.
Therefore the rectangle $AB$, $BC$ is equal to the rectangle $AC$, $CB$, together with the square on $BC$.

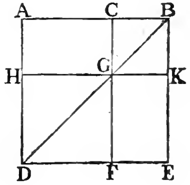
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 4. *THEOREM.*

*If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.*

Let the straight line $AB$ be divided into any two parts at the point $C$: the square on $AB$ shall be equal to the squares on $AC$, $CB$, together with twice the rectangle contained by $AC$, $CB$.

On $AB$ describe the square $ADEB$; [I. 46.
join $BD$; through $C$ draw $CGF$ parallel to $AD$ or $BE$, and through $G$ draw $HK$ parallel to $AB$ or $DE$. [I. 31.



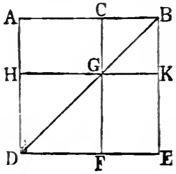

Then, because $CF$ is parallel to $AD$, and $BD$ falls on them, the exterior angle $CGB$ is equal to the interior and opposite angle $ADB$; [I. 29.
but the angle $ADB$ is equal to the angle $ABD$, [I. 5.
because $BA$ is equal to $AD$, being sides of a square;
therefore the angle $CGB$ is equal to the angle $CBG$; [*Ax.* 1.
and therefore the side $CG$ is equal to the side $CB$. [I. 6.
But $CB$ is also equal to $GK$, and $CG$ to $BK$; [I. 34.
therefore the figure $CGKB$ is equilateral.

It is likewise rectangular. For since $CG$ is parallel to $BK$, and $CB$ meets them, the angles $KBC$, $GCB$ are together equal to two right angles. [I. 29.

But $KBC$ is a right angle. [I. *Definition* 30

Therefore $GCB$ is a right angle. [*Axiom* 3.

And therefore also the angles $CGK$, $GKB$ opposite to these are right angles. [I. 34. and *Axiom* 1.




Therefore $CGKB$ is rectangular; and it has been shewn to be equilateral; therefore it is a square, and it is on the side $CB$.

For the same reason $HF$ is also a square, and it is on the side $HG$, which is equal to $AC$. [I. 34.

Therefore $HF$, $CK$ are the squares on $AC$, $CB$.

And because the complement $AG$ is equal to the complement $GE$; [I. 43.

and that $AG$ is the rectangle contained by $AC$, $CB$, for $CG$ is equal to $CB$;

therefore $GE$ is also equal to the rectangle $AC$, $CB$. [*Ax.* 1.

Therefore $AG$, $GE$ are equal to twice the rectangle $AC$, $CB$.

And $HF$, $CK$ are the squares on $AC$, $CB$.

Therefore the four figures $HF$, $CK$, $AG$, $GE$ are equal to the squares on $AC$, $CB$, together with twice the rectangle $AC$, $CB$.

But $HF$, $CK$, $AG$, $GE$ make up the whole figure $ADEB$, which is the square on $AB$.

Therefore the square on $AB$ is equal to the squares on $AC$, $CB$, together with twice the rectangle $AC$, $CB$.

Wherefore, *if a straight line* &c. Q.E.D.

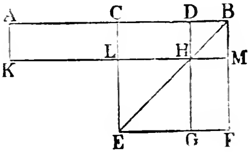
COROLLARY. From the demonstration it is manifest, that parallellograms about the diameter of a square are likewise squares.

## PROPOSITION 5. *THEOREM.*

*If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the*

*unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.*

Let the straight line $AB$ be divided into two equal parts at the point $C$, and into two unequal parts at the point $D$: the rectangle $AD, DB$, together with the square on $CD$, shall be equal to the square on $CB$.




On $CB$ describe the square $CEFB$; [I. 46.
join $BE$; through $D$ draw $DHG$ parallel to $CE$ or $BF$; through $H$ draw $KLM$ parallel to $CB$ or $EF$; and through $A$ draw $AK$ parallel to $CL$ or $BM$. [I. 31.

Then the complement $CH$ is equal to the complement $HF$; [I. 43.
to each of these add $DM$; therefore the whole $CM$ is equal to the whole $DF$. [*Axiom* 2.
But $CM$ is equal to $AL$, [I. 36.
because $AC$ is equal to $CB$. [*Hypothesis.*
Therefore also $AL$ is equal to $DF$. [*Axiom* 1.
To each of these add $CH$; therefore the whole $AH$ is equal to $DF$ and $CH$. [*Axiom* 2.
But $AH$ is the rectangle contained by $AD$, $DB$, for $DH$ is equal to $DB$; [II. 4, *Corollary.*
and $DF$ together with $CH$ is the gnomon $CMG$;
therefore the gnomon $CMG$ is equal to the rectangle $AD, DB$.
To each of these add $LG$, which is equal to the square on $CD$. [II. 4, *Corollary*, and I. 34.
Therefore the gnomon $CMG$, together with $LG$, is equal to the rectangle $AD, DB$, together with the square on $CD$. [*Ax.* 2.
But the gnomon $CMG$ and $LG$ make up the whole figure $CEFB$, which is the square on $CB$.
Therefore the rectangle $AD, DB$, together with the square on $CD$, is equal to the square on $CB$.

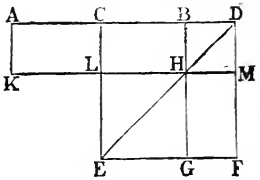
Wherefore, *if a straight line* &c. Q.E.D.

From this proposition it is manifest that the difference of the squares on two unequal straight lines $AC$, $CD$, is equal to the rectangle contained by their sum and difference.

PROPOSITION 6. *THEOREM.*

*If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.*

Let the straight line *AB* be bisected at the point *C*, and produced to the point *D*: the rectangle *AD, DB*, together with the square on *CB*, shall be equal to the square on *CD*.




On *CD* describe the square *CEFD*; [I. 46.
join *DE*; through *B* draw *BHG* parallel to *CE* or *DF*; through *H* draw *KLM* parallel to *AD* or *EF*; and through *A* draw *AK* parallel to *CL* or *DM*. [I. 31.

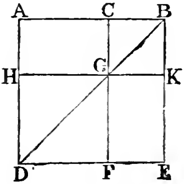
Then, because *AC* is equal to *CB*, [*Hypothesis.*
the rectangle *AL* is equal to the rectangle *CH*; [I. 36.
but *CH* is equal to *HF*; [I. 43.
therefore also *AL* is equal to *HF*. [*Axiom* 1.
To each of these add *CM*;
therefore the whole *AM* is equal to the gnomon *CMG*. [*Ax.* 2.
But *AM* is the rectangle contained by *AD, DB*, for *DM* is equal to *DB*. [II. 4, *Corollary.*
Therefore the rectangle *AD, DB* is equal to the gnomon *CMG*. [*Axiom* 1.
To each of these add *LG*, which is equal to the square on *CB*. [II. 4, *Corollary*, and I. 34.
Therefore the rectangle *AD, DB*, together with the square on *CB*, is equal to the gnomon *CMG* and the figure *LG*.
But the gnomon *CMG* and *LG* make up the whole figure *CEFD*, which is the square on *CD*.
Therefore the rectangle *AD, DB*, together with the square on *CB*, is equal to the square on *CD*.

Wherefore, *if a straight line* &c. Q.E.D.

## PROPOSITION 7. *THEOREM.*

*If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.*

Let the straight line $AB$ be divided into any two parts at the point $C$: the squares on $AB$, $BC$ shall be equal to twice the rectangle $AB$, $BC$, together with the square on $AC$.



On $AB$ describe the square $ADEB$, and construct the figure as in the preceding propositions.

Then $AG$ is equal to $GE$; [I. 43.

to each of these add $CK$;

therefore the whole $AK$ is equal to the whole $CE$;

therefore $AK$, $CE$ are double of $AK$.

But $AK$, $CE$ are the gnomon $AKF$, together with the square $CK$;

therefore the gnomon $AKF$, together with the square $CK$, is double of $AK$.

But twice the rectangle $AB$, $BC$ is double of $AK$, for $BK$ is equal to $BC$. [II. 4, *Corollary.*

Therefore the gnomon $AKF$, together with the square $CK$, is equal to twice the rectangle $AB$, $BC$.

To each of these equals add $HF$, which is equal to the square on $AC$. [II. 4, *Corollary*, and I. 34.

Therefore the gnomon $AKF$, together with the squares $CK$, $HF$, is equal to twice the rectangle $AB$, $BC$, together with the square on $AC$.

But the gnomon $AKF$ together with the squares $CK$, $HF$, make up the whole figure $ADEB$ and $CK$, which are the squares on $AB$ and $BC$.

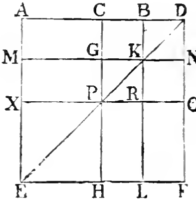
Therefore the squares on $AB$, $BC$, are equal to twice the rectangle $AB$, $BC$, together with the square on $AC$.

Wherefore, *if a straight line* &c. Q.E.D.

## PROPOSITION 8. *THEOREM.*

*If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.*

Let the straight line $AB$ be divided into any two parts at the point $C$: four times the rectangle $AB$, $BC$, together with the square on $AC$, shall be equal to the square on the straight line made up of $AB$ and $BC$ together.




Produce $AB$ to $D$, so that $BD$ may be equal to $CB$; [*Post.* 2. and I. 3.

on $AD$ describe the square $AEFD$;

and construct two figures such as in the preceding propositions.

Then, because $CB$ is equal to $BD$, [*Construction.*

and that $CB$ is equal to $GK$, and $BD$ to $KN$, [I. 34.

therefore $GK$ is equal to $KN$. [*Axiom* 1.

For the same reason $PR$ is equal to $RO$.

And because $CB$ is equal to $BD$, and $GK$ to $KN$, the rectangle $CK$ is equal to the rectangle $BN$, and the rectangle $GR$ to the rectangle $RN$. [I. 36.

But $CK$ is equal to $RN$, because they are the complements of the parallelogram $CO$; [I. 43.

therefore also $BN$ is equal to $GR$. [*Axiom* 1.

Therefore the four rectangles $BN$, $CK$, $GR$, $RN$ are equal to one another, and so the four are quadruple of one of them $CK$.

Again, because $CB$ is equal to $BD$, [*Construction.*

and that $BD$ is equal to $BK$, [II. 4, *Corollary.*

that is to $CG$, [I. 34.

and that $CB$ is equal to $GK$, [I. 34.

that is to $GP$; [II. 4, *Corollary.*

therefore $CG$ is equal to $GP$. [*Axiom* 1.

And because $CG$ is equal to $GP$, and $PR$ to $RO$, the rectangle $AG$ is equal to the rectangle $MP$, and the rectangle $PL$ to the rectangle $RF$. [I. 36.

But $MP$ is equal to $PL$, because they are the complements of the parallelogram $ML$; [I. 43.

therefore also $AG$ is equal to $RF$. [*Axiom* 1.

Therefore the four rectangles $AG$, $MP$, $PL$, $RF$ are equal to one another, and so the four are quadruple of one of them $AG$.

And it was shewn that the four $CK$, $BN$, $GR$ and $RN$ are quadruple of $CK$; therefore the eight rectangles which make up the gnomon $AOH$ are quadruple of $AK$.

And because $AK$ is the rectangle contained by $AB$, $BC$, for $BK$ is equal to $BC$;

therefore four times the rectangle $AB$, $BC$ is quadruple of $AK$.

But the gnomon $AOH$ was shewn to be quadruple of $AK$.

Therefore four times the rectangle $AB$, $BC$ is equal to the gnomon $AOH$. [*Axiom* 1.

To each of these add $XH$, which is equal to the square on $AC$. [II. 4, *Corollary*, and I. 34.

Therefore four times the rectangle $AB$, $BC$, together with the square on $AC$, is equal to the gnomon $AOH$ and the square $XH$.

But the gnomon $AOH$ and the square $XH$ make up the figure $AEFD$, which is the square on $AD$.

Therefore four times the rectangle $AB$, $BC$, together with the square on $AC$, is equal to the square on $AD$, that is to the square on the line made of $AB$ and $BC$ together.

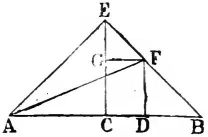
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 9. *THEOREM.*

*If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.*

Let the straight line $AB$ be divided into two equal parts at the point $C$, and into two unequal parts at the point $D$: the squares on $AD$, $DB$ shall be together double of the squares on $AC$, $CD$.

From the point $C$ draw $CE$ at right angles to $AB$, [I. 11. and make it equal to $AC$ or $CB$, [I. 3. and join $EA$, $EB$; through $D$ draw $DF$ parallel to $CE$, and through $F$ draw $FG$ parallel to $BA$; [I. 31. and join $AF$.




Then, because $AC$ is equal to $CE$, [*Construction.*

the angle $EAC$ is equal to the angle $AEC$. [I. 5.

And because the angle $ACE$ is a right angle, [*Construction.*

the two other angles $AEC$, $EAC$ are together equal to one right angle; [I. 32.

and they are equal to one another;

therefore each of them is half a right angle.

For the same reason each of the angles $CEB$, $EBC$ is half a right angle.

Therefore the whole angle $AEB$ is a right angle.

And because the angle $GEF$ is half a right angle, and the angle $EGF$ a right angle, for it is equal to the interior and opposite angle $ECB$; [I. 29.

therefore the remaining angle $EFG$ is half a right angle.

Therefore the angle $GEF$ is equal to the angle $EFG$, and the side $EG$ is equal to the side $GF$. [I. 6.

Again, because the angle at $B$ is half a right angle, and the

angle $FDB$ a right angle, for it is equal to the interior and opposite angle $ECB$; [I. 29.

therefore the remaining angle $BFD$ is half a right angle. Therefore the angle at $B$ is equal to the angle $BFD$, and the side $DF$ is equal to the side $DB$. [I. 6.

And because $AC$ is equal to $CE$, [*Construction.*

the square on $AC$ is equal to the square on $CE$;

therefore the squares on $AC$, $CE$ are double of the square on $AC$.

But the square on $AE$ is equal to the squares on $AC$, $CE$, because the angle $ACE$ is a right angle; [I. 47.

therefore the square on $AE$ is double of the square on $AC$.

Again, because $EG$ is equal to $GF$, [*Construction.*

the square on $EG$ is equal to the square on $GF$;

therefore the squares on $EG$, $GF$ are double of the square on $GF$.

But the square on $EF$ is equal to the squares on $EG$, $GF$, because the angle $EGF$ is a right angle; [I. 47.

therefore the square on $EF$ is double of the square on $GF$.

And $GF$ is equal to $CD$; [I. 34.

therefore the square on $EF$ is double of the square on $CD$.

But it has been shewn that the square on $AE$ is also double of the square on $AC$.

Therefore the squares on $AE$, $EF$ are double of the squares on $AC$, $CD$.

But the square on $AF$ is equal to the squares on $AE$, $EF$, because the angle $AEF$ is a right angle. [I. 47.

Therefore the square on $AF$ is double of the squares on $AC$, $CD$.

But the squares on $AD$, $DF$ are equal to the square on $AF$, because the angle $ADF$ is a right angle. [I. 47.

Therefore the squares on $AD$, $DF$ are double of the squares on $AC$, $CD$.

And $DF$ is equal to $DB$;

therefore the squares on $AD$, $DB$ are double of the squares on $AC$, $CD$.

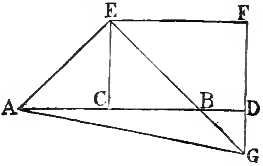
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 10. *THEOREM.*

*If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.*

Let the straight line $AB$ be bisected at $C$, and produced to $D$: the squares on $AD$, $DB$ shall be together double of the squares on $AC$, $CD$.

From the point $C$ draw $CE$ at right angles to $AB$, [I. 11. and make it equal to $AC$ or $CB$; [I. 3. and join $AE$, $EB$; through $E$ draw $EF$ parallel to $AB$, and through $D$ draw $DF$ parallel to $CE$. [I. 31.




Then because the straight line $EF$ meets the parallels $EC$, $FD$, the angles $CEF$, $EFD$ are together equal to two right angles; [I. 29.
and therefore the angles $BEF$, $EFD$ are together less than two right angles.
Therefore the straight lines $EB$, $FD$ will meet, if produced, towards $B$, $D$. [*Axiom* 12.
Let them meet at $G$, and join $AG$.

Then because $AC$ is equal to $CE$, [*Construction.*
the angle $CEA$ is equal to the angle $EAC$; [I. 5.
and the angle $ACE$ is a right angle; [*Construction.*
therefore each of the angles $CEA$. $EAC$ is half a right angle. [I. 32.
For the same reason each of the angles $CEB$, $EBC$ is half a right angle.
Therefore the angle $AEB$ is a right angle.

And because the angle $EBC$ is half a right angle, the angle $DBG$ is also half a right angle, for they are vertically opposite; [I. 15.
but the angle $BDG$ is a right angle, because it is equal to the alternate angle $DCE$; [I. 29.
therefore the remaining angle $DGB$ is half a right angle, [I. 32.

and is therefore equal to the angle $DBG$;
therefore also the side $BD$ is equal to the side $DG$. [I. 6.
Again, because the angle $EGF$ is half a right angle, and the angle at $F$ a right angle, for it is equal to the opposite angle $ECD$; [I. 34.
therefore the remaining angle $FEG$ is half a right angle, [I. 32.
and is therefore equal to the angle $EGF$;
therefore also the side $GF$ is equal to the side $FE$. [I. 6.

And because $EC$ is equal to $CA$, the square on $EC$ is equal to the square on $CA$;
therefore the squares on $EC$, $CA$ are double of the square on $CA$.
But the square on $AE$ is equal to the squares on $EC$, $CA$. [I. 47.
Therefore the square on $AE$ is double of the square on $AC$.
Again, because $GF$ is equal to $FE$, the square on $GF$ is equal to the square on $FE$;
therefore the squares on $GF$, $FE$ are double of the square on $FE$.
But the square on $EG$ is equal to the squares on $GF$, $FE$. [I. 47.
Therefore the square on $EG$ is double of the square on $FE$.
And $FE$ is equal to $CD$; [I. 34.
therefore the square on $EG$ is double of the square on $CD$.
But it has been shewn that the square on $AE$ is double of the square on $AC$.
Therefore the squares on $AE$, $EG$ are double of the squares on $AC$, $CD$.

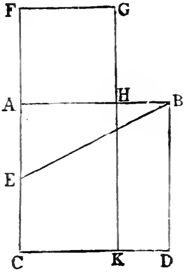
But the square on $AG$ is equal to the squares on $AE$, $EG$. [I. 47.
Therefore the square on $AG$ is double of the squares on $AC$, $CD$.
But the squares on $AD$, $DG$ are equal to the square on $AG$. [I. 47.
Therefore the squares on $AD$, $DG$ are double of the squares on $AC$, $CD$.
And $DG$ is equal to $DB$;
therefore the squares on $AD$, $DB$ are double of the squares on $AC$, $CD$.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 11. *PROBLEM.*

*To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.*

Let $AB$ be the given straight line: it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.




On $AB$ describe the square $ABDC$; [I. 46.

bisect $AC$ at $E$; [I. 10.

join $BE$; produce $CA$ to $F$, and make $EF$ equal to $EB$; [I. 3.

and on $AF$ describe the square $AFGH$. [I. 46.

$AB$ shall be divided at $H$ so that the rectangle $AB$. $BH$ is equal to the square on $AH$.

Produce $GH$ to $K$.

Then, because the straight line $AC$ is bisected at $E$, and produced to $F$, the rectangle $CF$, $FA$, together with the square on $AE$, is equal to the square on $EF$. [II. 6.

But $EF$ is equal to $EB$. [*Construction.*

Therefore the rectangle $CF$, $FA$, together with the square on $AE$, is equal to the square on $EB$.

But the square on $EB$ is equal to the squares on $AE$, $AB$, because the angle $EAB$ is a right angle. [I. 47.

Therefore the rectangle $CF$, $FA$, together with the square on $AE$, is equal to the squares on $AE$, $AB$.

Take away the square on $AE$, which is common to both; therefore the remainder, the rectangle $CF$, $FA$, is equal to the square on $AB$. [*Axiom* 3.

But the figure $FK$ is the rectangle contained by $CF$, $FA$, for $FG$ is equal to $FA$:

and $AD$ is the square on $AB$;

therefore $FK$ is equal to $AD$.

Take away the common part $AK$, and the remainder $FH$ is equal to the remainder $HD$. [*Axiom* 3.

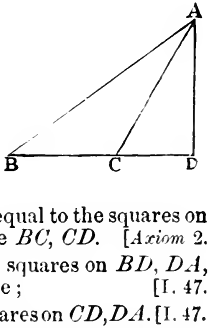
But *HD* is the rectangle contained by *AB*, *BH*, for *AB* is equal to *BD*;
and *FH* is the square on *AH*;
therefore the rectangle *AB*, *BH* is equal to the square on *AH*.

Wherefore *the straight line AB is divided at H, so that the rectangle AB, BH is equal to the square on AH.* Q.E.F.

## PROPOSITION 12. *THEOREM.*

*In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.*

Let *ABC* be an obtuse-angled triangle, having the obtuse angle *ACB*, and from the point *A* let *AD* be drawn perpendicular to *BC* produced: the square on *AB* shall be greater than the squares on *AC*, *CB*, by twice the rectangle *BC*, *CD*.




Because the straight line *BD* is divided into two parts at the point *C*, the square on *BD* is equal to the squares on *BC*, *CD*, and twice the rectangle *BC*, *CD*. [II. 4.

To each of these equals add the square on *DA*.

Therefore the squares on *BD*, *DA* are equal to the squares on *BC*, *CD*, *DA*, and twice the rectangle *BC*, *CD*. [*Axiom* 2.

But the square on *BA* is equal to the squares on *BD*, *DA*, because the angle at *D* is a right angle; [I. 47.

and the square on *CA* is equal to the squares on *CD*, *DA*. [I. 47.

Therefore the square on *BA* is equal to the squares on *BC*, *CA*, and twice the rectangle *BC*, *CD*;

that is, the square on *BA* is greater than the squares on *BC*, *CA* by twice the rectangle *BC*, *CD*.

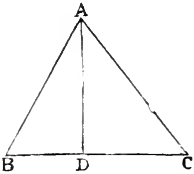
Wherefore, *in obtuse-angled triangles* &c. Q.E.D.

## PROPOSITION 13. *THEOREM.*

*In every triangle, the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle*

Let *ABC* be any triangle, and the angle at *B* an acute angle; and on *BC* one of the sides containing it, let fall the perpendicular *AD* from the opposite angle: the square on *AC*, opposite to the angle *B*, shall be less than the squares on *CB*, *BA*, by twice the rectangle *CB*, *BD*.

First, let *AD* fall within the triangle *ABC*.




Then, because the straight line *CB* is divided into two parts at the point *D*, the squares on *CB*, *BD* are equal to twice the rectangle contained by *CB*, *BD* and the square on *CD*. [II. 7.

To each of these equals add the square on *DA*.

Therefore the squares on *CB*, *BD*, *DA* are equal to twice the rectangle *CB*, *BD* and the squares on *CD*, *DA*. [*Ax.* 2.

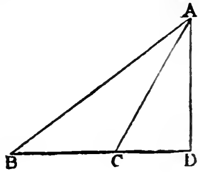
But the square on *AB* is equal to the squares on *BD*, *DA*, because the angle *BDA* is a right angle; [I. 47.

and the square on *AC* is equal to the squares on *CD*, *DA*. [I. 47.

Therefore the squares on *CB*, *BA* are equal to the square on *AC* and twice the rectangle *CB*, *BD*;

that is, the square on *AC* alone is less than the squares on *CB*, *BA* by twice the rectangle *CB*, *BD*.

Secondly, let *AD* fall without the triangle *ABC*.




Then because the angle at *D* is a right angle, [*Construction.*

the angle *ACB* is greater than a right angle; [I. 16.

and therefore the square on $AB$ is equal to the squares on $AC$, $CB$, and twice the rectangle $BC$, $CD$. [II. 12.

To each of these equals add the square on $BC$.

Therefore the squares on $AB$, $BC$ are equal to the square on $AC$, and twice the square on $BC$, and twice the rectangle $BC$, $CD$. [*Axiom* 2.

But because $BD$ is divided into two parts at $C$, the rectangle $DB$, $BC$ is equal to the rectangle $BC$, $CD$ and the square on $BC$; [II. 3.

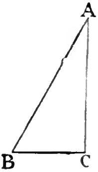
and the doubles of these are equal,

that is, twice the rectangle $DB$, $BC$ is equal to twice the rectangle $BC$, $CD$ and twice the square on $BC$.

Therefore the squares on $AB$, $BC$ are equal to the square on $AC$, and twice the rectangle $DB$, $BC$;

that is, the square on $AC$ alone is less than the squares on $AB$, $BC$ by twice the rectangle $DB$, $BC$.

Lastly, let the side $AC$ be perpendicular to $BC$.




Then $BC$ is the straight line between the perpendicular and the acute angle at $B$;

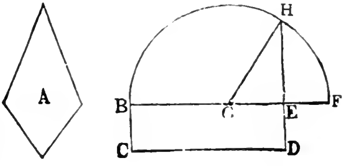
and it is manifest, that the squares on $AB$, $BC$ are equal to the square on $AC$, and twice the square on $BC$. [I. 47 and *Ax.* 2.

Wherefore, *in every triangle* &c. Q.E.D.

## PROPOSITION 14. *PROBLEM.*

*To describe a square that shall be equal to a given rectilineal figure.*

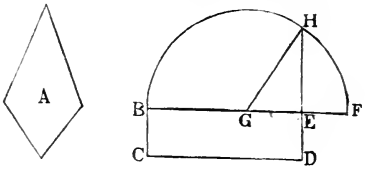
Let $A$ be the given rectilineal figure: it is required to describe a square that shall be equal to $A$.




Describe the rectangular parallelogram $BCDE$ equal to the rectilineal figure $A$. [I. 45.

Then if the sides of it, $BE$, $ED$ are equal to one another, it is a square, and what was required is now done.

But if they are not equal, produce one of them $BE$ to $F$, make $EF$ equal to $ED$, [I. 3. and bisect $BF$ at $G$; [I. 10. from the centre $G$, at the distance $GB$, or $GF$, describe the semicircle $BHF$, and produce $DE$ to $H$.




The square described on $EH$ shall be equal to the given rectilineal figure $A$.

Join $GH$. Then, because the straight line $BF$ is divided into two equal parts at the point $G$, and into two unequal parts at the point $E$, the rectangle $BE$, $EF$, together with the square on $GE$, is equal to the square on $GF$. [II. 5.
But $GF$ is equal to $GH$.
Therefore the rectangle $BE$, $EF$, together with the square on $GE$, is equal to the square on $GH$.
But the square on $GH$ is equal to the squares on $GE$, $EH$; [I. 47.
therefore the rectangle $BE$, $EF$, together with the square on $GE$, is equal to the squares on $GE$, $EH$.
Take away the square on $GE$, which is common to both;
therefore the rectangle $BE$, $EF$ is equal to the square on $EH$. [*Axiom* 3.
But the rectangle contained by $BE$, $EF$ is the parallelogram $BD$,
because $EF$ is equal to $ED$. [*Construction.*
Therefore $BD$ is equal to the square on $EH$.
But $BD$ is equal to the rectilineal figure $A$. [*Construction.*
Therefore the square on $EH$ is equal to the rectilineal figure $A$.

Wherefore *a square has been made equal to the given rectilineal figure A, namely, the square described on EH.* Q.E.F.

# *BOOK III.*

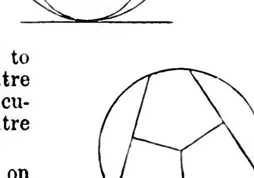
## DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.

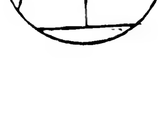
2. A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it.

3. Circles are said to touch one another, which meet but do not cut one another.



4. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

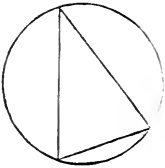
5. And the straight line on which the greater perpendicular falls, is said to be farther from the centre.



6. A segment of a circle is the figure contained by a straight line and the circumference it cuts off.

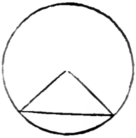
7. The angle of a segment is that which is contained by the straight line and the circumference.

8. An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.



9. And an angle is said to insist or stand on the circumference intercepted between the straight lines which contain the angle.

10. A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



11. Similar segments of circles are those in which the angles are equal, or which contain equal angles.

[*Note.* In the following propositions, whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that the lines are drawn to the circumference.

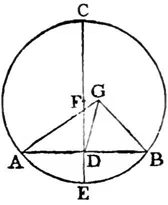
Any portion of the circumference is called an *arc.*]

## PROPOSITION 1. *PROBLEM.*

*To find the centre of a given circle.*

Let $ABC$ be the given circle: it is required to find its centre.

Draw within it any straight line $AB$, and bisect $AB$ at $D$; [I. 10.
from the point $D$ draw $DC$ at right angles to $AB$; [I. 11.
produce $CD$ to meet the circumference at $E$, and bisect $CE$ at $F$. [I. 10.
The point $F$ shall be the centre of the circle $ABC$.




For if $F$ be not the centre, if possible, let $G$ be the centre; and join $GA$, $GD$, $GB$.
Then, because $DA$ is equal to $DB$, [*Construction*.
and $DG$ is common to the two triangles $ADG$, $BDG$;
the two sides $AD$, $DG$ are equal to the two sides $BD$, $DG$, each to each;
and the base $GA$ is equal to the base $GB$, because they are drawn from the centre $G$; [I. *Definition* 15.
therefore the angle $ADG$ is equal to the angle $BDG$. [I. 8.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [I. *Definition* 10.
therefore the angle $BDG$ is a right angle.
But the angle $BDF$ is also a right angle. [*Construction*.
Therefore the angle $BDG$ is equal to the angle $BDF$, [*Ax.* 11.
the less to the greater; which is impossible.
Therefore $G$ is not the centre of the circle $ABC$.

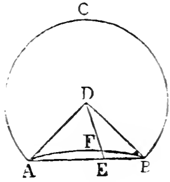
In the same manner it may be shewn that no other point out of the line $CE$ is the centre;
and since $CE$ is bisected at $F$, any other point in $CE$ divides it into unequal parts, and cannot be the centre.
Therefore no point but $F$ is the centre;
that is, $F$ is the centre of the circle $ABC$:
*which was to be found.*

Corollary. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the straight line which bisects the other.

PROPOSITION 2. *THEOREM.*

*If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.*

Let $ABC$ be a circle, and $A$ and $B$ any two points in the circumference: the straight line drawn from $A$ to $B$ shall fall within the circle.




For if it do not, let it fall, if possible, without, as $AEB$. Find $D$ the centre of the circle $ABC$; [III. 1.
and join $DA$, $DB$; in the arc $AB$ take any point $F$, join $DF$, and produce it to meet the straight line $AB$ at $E$.

Then, because $DA$ is equal to $DB$, [I. *Definition* 15.
the angle $DAB$ is equal to the angle $DBA$. [I. 5.

And because $AE$, a side of the triangle $DAE$, is produced to $B$, the exterior angle $DEB$ is greater than the interior opposite angle $DAE$. [I. 16.

But the angle $DAE$ was shewn to be equal to the angle $DBE$; therefore the angle $DEB$ is greater than the angle $DBE$.

But the greater angle is subtended by the greater side; [I. 19.
therefore $DB$ is greater than $DE$.

But $DB$ is equal to $DF$; [I. *Definition* 15.
therefore $DF$ is greater than $DE$, the less than the greater; which is impossible.

Therefore the straight line drawn from $A$ to $B$ does not fall without the circle.

In the same manner it may be shewn that it does not fall on the circumference.

Therefore it falls within the circle.

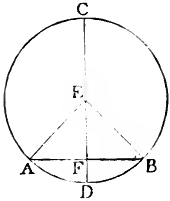
Wherefore, *if any two points* &c. Q.E.D.

PROPOSITION 3. *THEOREM.*

*If a straight line drawn through the centre of a circle, bisect a straight line in it which does not pass through the

*centre, it shall cut it at right angles; and if it cut it at right angles it shall bisect it.*

Let $ABC$ be a circle; and let $CD$, a straight line drawn through the centre, bisect any straight line $AB$, which does not pass through the centre, at the point $F$: $CD$ shall cut $AB$ at right angles.




Take $E$ the centre of the circle; and join $EA$, $EB$. [III. 1.

Then, because $AF$ is equal to $FB$, [*Hypothesis.*
and $FE$ is common to the two triangles $AFE$, $BFE$;
the two sides $AF$, $FE$ are equal to the two sides $BF$, $FE$, each to each;
and the base $EA$ is equal to the base $EB$; [I. *Def.* 15.
therefore the angle $AFE$ is equal to the angle $BFE$. [I. 8.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [I. *Definition* 10.
therefore each of the angles $AFE$, $BFE$ is a right angle.
Therefore the straight line $CD$, drawn through the centre, bisecting another $AB$ which does not pass through the centre, also cuts it at right angles.

But let $CD$ cut $AB$ at right angles: $CD$ shall also bisect $AB$; that is, $AF$ shall be equal to $FB$.

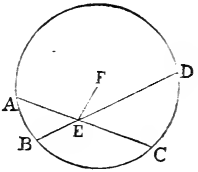
The same construction being made, because $EA$, $EB$, drawn from the centre, are equal to one another, [I. *Def.* 15.
the angle $EAF$ is equal to the angle $EBF$. [I. 5.
And the right angle $AFE$ is equal to the right angle $BFE$.
Therefore in the two triangles $EAF$, $EBF$, there are two angles in the one equal to two angles in the other, each to each;
and the side $EF$, which is opposite to one of the equal angles in each, is common to both;
therefore their other sides are equal; [I. 26.
therefore $AF$ is equal to $FB$.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 4. *THEOREM.*

*If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect one another.*

Let $ABCD$ be a circle, and $AC$, $BD$ two straight lines in it, which cut one another at the point $E$, and do not both pass through the centre: $AC$, $BD$ shall not bisect one another.




If one of the straight lines pass through the centre it is plain that it cannot be bisected by the other which does not pass through the centre.

But if neither of them pass through the centre, if possible, let $AE$ be equal to $EC$, and $BE$ equal to $ED$.

Take $F$ the centre of the circle [III. 1.

and join $EF$.

Then, because $FE$, a straight line drawn through the centre, bisects another straight line $AC$ which does not pass through the centre; [*Hypothesis.*

$FE$ cuts $AC$ at right angles; [III. 3.

therefore the angle $FEA$ is a right angle.

Again, because the straight line $FE$ bisects the straight line $BD$, which does not pass through the centre, [*Hyp.*

$FE$ cuts $BD$ at right angles; [III. 3.

therefore the angle $FEB$ is a right angle.

But the angle $FEA$ was shewn to be a right angle;

therefore the angle $FEA$ is equal to the angle $FEB$, [*Ax.* 11.

the less to the greater; which is impossible.

Therefore $AC$, $BD$ do not bisect each other.

Wherefore, *if in a circle* &c. Q.E.D.

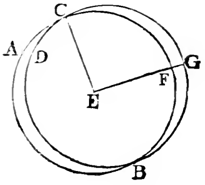
PROPOSITION 5. *THEOREM.*

*If two circles cut one another, they shall not have the same centre.*

Let the two circles $ABC$, $CDG$ cut one another at the

points $B$, $C$: they shall not have the same centre.

For, if it be possible, let $E$ be their centre; join $EC$, and draw any straight line $EFG$ meeting the circumferences at $F$ and $G$.




Then, because $E$ is the centre of the circle $ABC$, $EC$ is equal to $EF$. [I. *Definition* 15.

Again, because $E$ is the centre of the circle $CDG$, $EC$ is equal to $EG$. [I. *Definition* 15.

But $EC$ was shewn to be equal to $EF$;

therefore $EF$ is equal to $EG$, [*Axiom* 1.

the less to the greater; which is impossible.

Therefore $E$ is not the centre of the circles $ABC$, $CDG$.

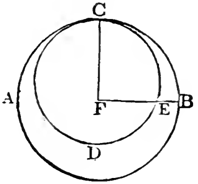
Wherefore, *if two circles* &c. Q.E.D.

PROPOSITION 6. *THEOREM.*

*If two circles touch one another internally, they shall not have the same centre.*

Let the two circles $ABC$, $CDE$ touch one another internally at the point $C$: they shall not have the same centre.

For, if it be possible, let $F$ be their centre; join $FC$, and draw any straight line $FEB$, meeting the circumferences at $E$ and $B$.




Then, because $F$ is the centre of the circle $ABC$, $FC$ is equal to $FB$. [I. *Def.* 15.

Again, because $F$ is the centre of the circle $CDE$, $FC$ is equal to $FE$. [I. *Definition* 15.

But $FC$ was shewn to be equal to $FB$;

therefore $FE$ is equal to $FB$, [*Axiom* 1.

the less to the greater; which is impossible.

Therefore $F$ is not the centre of the circles $ABC$, $CDE$.

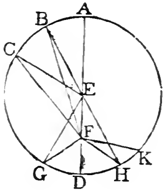
Wherefore, *if two circles* &c. Q.E.D.

PROPOSITION 7. *THEOREM.*

*If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference, the greatest is that in which the centre is, and the other part of the diameter is the least; and, of any others, that which is nearer to the straight line which passes through the centre, is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let $ABCD$ be a circle and $AD$ its diameter, in which let any point $F$ be taken which is not the centre; let $E$ be the centre: of all the straight lines $FB$, $FC$, $FG$, &c. that can be drawn from $F$ to the circumference, $FA$, which passes through $E$, shall be the greatest, and $FD$, the other part of the diameter $AD$, shall be the least; and of the others $FB$ shall be greater than $FC$, and $FC$ than $FG$.

Join $BE$, $CE$, $GE$.




Then, because any two sides of a triangle are greater than the third side, [I. 20.

therefore $BE$, $EF$ are greater than $BF$.

But $BE$ is equal to $AE$; [I. *Def.* 15.

therefore $AE$, $EF$ are greater than $BF$,

that is, $AF$ is greater than $BF$.

Again, because $BE$ is equal to $CE$, [I. *Definition* 15.

and $EF$ is common to the two triangles $BEF$, $CEF$;

the two sides $BE$, $EF$ are equal to the two sides $CE$, $EF$, each to each;

but the angle $BEF$ is greater than the angle $CEF$;

therefore the base $FB$ is greater than the base $FC$. [I. 24.

In the same manner it may be shewn that $FC$ is greater than $FG$.

Again, because $GF$, $FE$ are greater than $EG$, [I. 20.

and that $EG$ is equal to $ED$; [I. *Definition* 15.
therefore $GF$, $FE$ are greater than $ED$.
Take away the common part $FE$, and the remainder $GF$ is greater than the remainder $FD$.

Therefore $FA$ is the greatest, and $FD$ the least of all the straight lines from $F$ to the circumference; and $FB$ is greater than $FC$, and $FC$ than $FG$.

Also, there can be drawn two equal straight lines from the point $F$ to the circumference, one on each side of the shortest line $FD$.

For, at the point $E$, in the straight line $EF$, make the angle $FEH$ equal to the angle $FEG$, [I. 23.
and join $FH$.

Then, because $EG$ is equal to $EH$, [I. *Definition* 15.
and $EF$ is common to the two triangles $GEF$, $HEF$;
the two sides $EG$, $EF$ are equal to the two sides $EH$, $EF$, each to each;
and the angle $GEF$ is equal to the angle $HEF$; [*Constr.*
therefore the base $FG$ is equal to the base $FH$. [I. 4.

But, besides $FH$, no other straight line can be drawn from $F$ to the circumference, equal to $FG$.

For, if it be possible, let $FK$ be equal to $FG$.
Then, because $FK$ is equal to $FG$, [*Hypothesis.*
and $FH$ is also equal to $FG$,
therefore $FH$ is equal to $FK$; [*Axiom* 1.
that is, a line nearer to that which passes through the centre is equal to a line which is more remote;
which is impossible by what has been already shewn.

Wherefore, *if any point be taken* &c. Q.E.D.

## PROPOSITION 8. *THEOREM.*

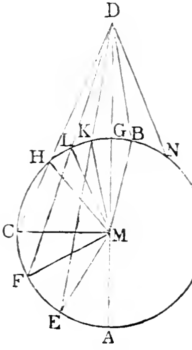
*If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre; of those which fall on the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but of those which fall on the*

*convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let $ABC$ be a circle, and $D$ any point without it, and from $D$ let the straight lines $DA$, $DE$, $DF$, $DC$ be drawn to the circumference, of which $DA$ passes through the centre: of those which fall on the concave circumference $AEFC$, the greatest shall be $DA$ which passes through the centre, and the nearer to it shall be greater than the more remote, namely, $DE$ greater than $DF$, and $DF$ greater than $DC$; but of those which fall on the convex circumference $GKLH$, the least shall be $DG$ between the point $D$ and the diameter $AG$, and the nearer to it shall be less than the more remote, namely, $DK$ less than $DL$, and $DL$ less than $DH$.

Take $M$, the centre of the circle $ABC$, [III. 1.

and join $ME$, $MF$, $MC$, $MH$, $ML$, $MK$.




Then, because any two sides of a triangle are greater than the third side, [I. 20.

therefore $EM$, $MD$ are greater than $ED$.

But $EM$ is equal to $AM$; [I. *Def.* 15.

therefore $AM$, $MD$ are greater than $ED$,

that is, $AD$ is greater than $ED$.

Again, because $EM$ is equal to $FM$,

and $MD$ is common to the two triangles $EMD$, $FMD$;

the two sides $EM$, $MD$ are equal to the two sides $FM$, $MD$, each to each;

but the angle $EMD$ is greater than the angle $FMD$;

therefore the base $ED$ is greater than the base $FD$. [I. 24.

In the same manner it may be shewn that $FD$ is greater than $CD$.

Therefore $DA$ is the greatest, and $DE$ greater than $DF$, and $DF$ greater than $DC$.

Again, because $MK$, $KD$ are greater than $MD$, [I. 20.
and $MK$ is equal to $MG$, [I. *Definition* 15.
the remainder $KD$ is greater than the remainder $GD$,
that is, $GD$ is less than $KD$.

And because $MLD$ is a triangle, and from the points $M$, $D$, the extremities of its side $MD$, the straight lines $MK$, $DK$ are drawn to the point $K$ within the triangle, therefore $MK$, $KD$ are less than $ML$, $LD$; [I. 21.
and $MK$ is equal to $ML$; [I. *Definition* 15.
therefore the remainder $KD$ is less than the remainder $LD$.

In the same manner it may be shewn that $LD$ is less than $HD$.
Therefore $DG$ is the least, and $DK$ less than $DL$, and $DL$ less than $DH$.

Also, there can be drawn two equal straight lines from the point $D$ to the circumference, one on each side of the least line.

For, at the point $M$, in the straight line $MD$, make the angle $DMB$ equal to the angle $DMK$, [I. 23.
and join $DB$.

Then, because $MK$ is equal to $MB$,
and $MD$ is common to the two triangles $KMD$, $BMD$;
the two sides $KM$, $MD$ are equal to the two sides $BM$, $MD$, each to each;
and the angle $DMK$ is equal to the angle $DMB$; [*Constr.*
therefore the base $DK$ is equal to the base $DB$. [I. 4.

But, besides $DB$, no other straight line can be drawn from $D$ to the circumference, equal to $DK$.

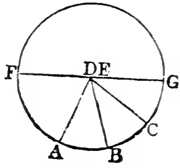
For, if it be possible, let $DN$ be equal to $DK$.
Then, because $DN$ is equal to $DK$,
and $DB$ is also equal to $DK$,
therefore $DB$ is equal to $DN$; [*Axiom* 1.
that is, a line nearer to the least is equal to one which is more remote;
which is impossible by what has been already shewn.

Wherefore, *if any point be taken* &c. Q.E.D.

## PROPOSITION 9. *THEOREM.*

*If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.*

Let the point $D$ be taken within the circle $ABC$, from which to the circumference there fall more than two equal straight lines, namely $DA$, $DB$, $DC$: the point $D$ shall be the centre of the circle.




For, if not, let $E$ be the centre; join $DE$ and produce it both ways to meet the circumference at $F$ and $G$; then $FG$ is a diameter of the circle.

Then, because in $FG$, a diameter of the circle $ABC$, the point $D$ is taken, which is not the centre, $DG$ is the greatest straight line from $D$ to the circumference, and $DC$ is greater than $DB$, and $DB$ greater than $DA$; [III. 7.

but they are likewise equal, by hypothesis;

which is impossible.

Therefore $E$ is not the centre of the circle $ABC$.

In the same manner it may be shewn that any other point than $D$ is not the centre;

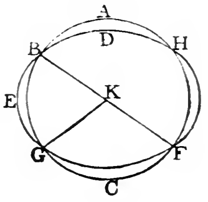
therefore $D$ is the centre of the circle $ABC$.

Wherefore, *if a point be taken* &c. Q.E.D.

## PROPOSITION 10. *THEOREM.*

*One circumference of a circle cannot cut another at more than two points.*

If it be possible, let the circumference $ABC$ cut the circumference $DEF$ at more than two points, namely, at the points $B$, $G$, $F$.




Take $K$, the centre of the circle $ABC$, [III. 1.

and join $KB$, $KG$, $KF$.

Then, because $K$ is the centre of the circle $ABC$,

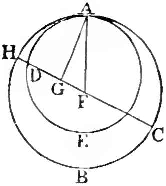
therefore $KB$, $KG$, $KF$ are all equal to each other. [I. *Def.* 15.
And because within the circle $DEF$, the point $K$ is taken, from which to the circumference $DEF$ fall more than two equal straight lines $KB$, $KG$, $KF$, therefore $K$ is the centre of the circle $DEF$. [III. 9.
But $K$ is also the centre of the circle $ABC$. [*Construction.*
Therefore the same point is the centre of two circles which cut one another;
which is impossible. [III. 5.

Wherefore, *one circumference* &c. Q.E.D.

PROPOSITION 11. *THEOREM.*

*If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.*

Let the two circles $ABC$, $ADE$ touch one another internally at the point $A$; and let $F$ be the centre of the circle $ABC$, and $G$ the centre of the circle $ADE$: the straight line which joins the centres $F$, $G$, being produced, shall pass through the point $A$.




For, if not, let it pass otherwise, if possible, as $FGDH$, and join $AF$, $AG$.

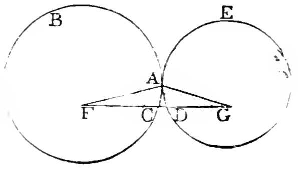
Then, because $AG$, $GF$ are greater than $AF$, [I. 20.
and $AF$ is equal to $HF$, [I. *Def.* 15.
therefore $AG$, $GF$, are greater than $HF$.
Take away the common part $GF$;
therefore the remainder $AG$ is greater than the remainder $HG$.
But $AG$ is equal to $DG$. [I. *Definition* 15.
Therefore $DG$ is greater than $HG$, the less than the greater; which is impossible.
Therefore the straight line which joins the points $F$, $G$, being produced, cannot pass otherwise than through the point $A$,
that is, it must pass through $A$.

Wherefore, *if two circles* &c. Q.E.D.

## PROPOSITION 12. *THEOREM.*

*If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.*

Let the two circles $ABC$, $ADE$ touch one another externally at the point $A$; and let $F$ be the centre of the circle $ABC$, and $G$ the centre of the circle $ADE$: the straight line which joins the points $F$, $G$, shall pass through the point $A$.




For, if not, let it pass otherwise, if possible, as $FCDG$, and join $FA$, $AG$.

Then, because $F$ is the centre of the circle $ABC$, $FA$ is equal to $FC$; [I. *Def.* 15.
and because $G$ is the centre of the circle $ADE$, $GA$ is equal to $GD$;
therefore $FA$, $AG$ are equal to $FC$, $DG$. [*Axiom* 2.
Therefore the whole $FG$ is greater than $FA$, $AG$.
But $FG$ is also less than $FA$, $AG$; [I. 20.
which is impossible.
Therefore the straight line which joins the points $F$, $G$ cannot pass otherwise than through the point $A$,
that is, it must pass through $A$.

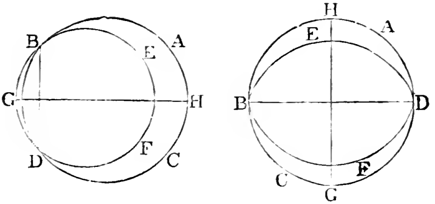
Wherefore, *if two circles* &c. Q.E.D.

## PROPOSITION 13. *THEOREM.*

*One circle cannot touch another at more points than one, whether it touches it on the inside or outside.*

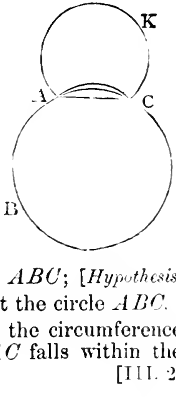
For, if it be possible, let the circle $EBF$ touch the circle $ABC$ at more points than one; and first on the inside, at the points $B$, $D$. Join $BD$, and draw $GH$ bisecting $BD$ at right angles. [I. 10, 11.

Then, because the two points $B$, $D$ are in the circumference of each of the circles, the straight line $BD$ falls within each of them; [III. 2.




and therefore the centre of each circle is in the straight line $GH$ which bisects $BD$ at right angles; [III. 1, *Corol.*
therefore $GH$ passes through the point of contact. [III. 11.
But $GH$ does not pass through the point of contact, because the points $B$, $D$ are out of the line $GH$;
which is absurd.

Therefore one circle cannot touch another on the inside at more points than one.




Nor can one circle touch another on the outside at more points than one.

For, if it be possible, let the circle $ACK$ touch the circle $ABC$ at the points $A$, $C$. Join $AC$.

Then, because the two points $A$, $C$ are in the circumference of the circle $ACK$, the straight line $AC$ which joins them, falls within the circle $ACK$; [III. 2.
but the circle $ACK$ is without the circle $ABC$; [*Hypothesis.*
therefore the straight line $AC$ is without the circle $ABC$.
But because the two points $A$, $C$ are in the circumference of the circle $ABC$, the straight line $AC$ falls within the circle $ABC$; [III. 2.
which is absurd.

Therefore one circle cannot touch another on the outside at more points than one.

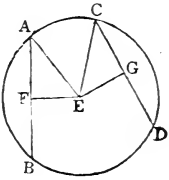
And it has been shewn that one circle cannot touch another on the inside at more points than one.

Wherefore, *one circle* &c. Q.E.D.

PROPOSITION 14. *THEOREM.*

*Equal straight lines in a circle are equally distant from the centre: and those which are equally distant from the centre are equal to one another.*

Let the straight lines $AB$, $CD$ in the circle $ABDC$, be equal to one another: they shall be equally distant from the centre.




Take $E$, the centre of the circle $ABDC$; [III. 1.

and from $E$ draw $EF$, $EG$ perpendiculars to $AB$, $CD$; [I. 12.

and join $EA$, $EC$.

Then, because the straight line $EF$, passing through the centre, cuts the straight line $AB$, which does not pass through the centre, at right angles, it also bisects it; [III. 3.

therefore $AF$ is equal to $FB$, and $AB$ is double of $AF$.

For the like reason $CD$ is double of $CG$.

But $AB$ is equal to $CD$; [*Hypothesis.*

therefore $AF$ is equal to $CG$. [*Axiom* 7.

And because $AE$ is equal to $CE$, [I. *Definition* 15.

the square on $AE$ is equal to the square on $CE$.

But the squares on $AF$, $FE$ are equal to the square on $AE$, because the angle $AFE$ is a right angle; [I. 47.

and for the like reason the squares on $CG$, $GE$ are equal to the square on $CE$;

therefore the squares on $AF$, $FE$ are equal to the squares on $CG$, $GE$. [*Axiom* 1.

But the square on $AF$ is equal to the square on $CG$, because $AF$ is equal to $CG$;

therefore the remaining square on $FE$ is equal to the remaining square on $GE$; [*Axiom* 3.

and therefore the straight line $EF$ is equal to the straight line $EG$.

But straight lines in a circle are said to be equally distant

from the centre, when the perpendiculars drawn to them from the centre are equal; [III. *Definition* 4.

therefore $AB$, $CD$ are equally distant from the centre.

Next, let the straight lines $AB$, $CD$ be equally distant from the centre, that is, let $EF$ be equal to $EG$: $AB$ shall be equal to $CD$.

For, the same construction being made, it may be shewn, as before, that $AB$ is double of $AF$, and $CD$ double of $CG$, and that the squares on $EF$, $FA$ are equal to the squares on $EG$, $GC$;

but the square on $EF$ is equal to the square on $EG$, because $EF$ is equal to $EG$; [*Hypothesis*.

therefore the remaining square on $FA$ is equal to the remaining square on $GC$, [*Axiom* 3.

and therefore the straight line $AF$ is equal to the straight line $CG$.

But $AB$ was shewn to be double of $AF$, and $CD$ double of $CG$.

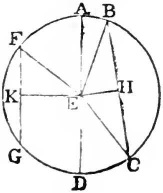
Therefore $AB$ is equal to $CD$. [*Axiom* 6.

Wherefore, *equal straight lines* &c. Q.E.D.

PROPOSITION 15. *THEOREM.*

*The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.*

Let $ABCD$ be a circle, of which $AD$ is a diameter, and $E$ the centre; and let $BC$ be nearer to the centre than $FG$: $AD$ shall be greater than any straight line $BC$ which is not a diameter, and $BC$ shall be greater than $FG$.




From the centre $E$ draw $EH$, $EK$ perpendiculars to $BC$, $FG$, [I. 12.

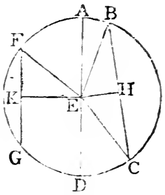
and join $EB$, $EC$, $EF$.

Then, because $AE$ is equal to $BE$, and $ED$ to $EC$, [I. *Def.* 15.

therefore $AD$ is equal to $BE$, $EC$; [*Axiom* 2.

but $BE$, $EC$ are greater than $BC$; [I. 20.
therefore also $AD$ is greater than $BC$.

And, because $BC$ is nearer to the centre than $FG$, [*Hypothesis.*
$EH$ is less than $EK$. [III. *Def.* 5.
Now it may be shewn, as in the preceding proposition, that $BC$ is double of $BH$, and $FG$ double of $FK$, and that the squares on $EH$, $HB$ are equal to the squares on $EK$, $KF$.




But the square on $EH$ is less than the square on $EK$, because $EH$ is less than $EK$;
therefore the square on $HB$ is greater than the square on $KF$;
and therefore the straight line $BH$ is greater than the straight line $FK$;
and therefore $BC$ is greater than $FG$.

Next, let $BC$ be greater than $FG$: $BC$ shall be nearer to the centre than $FG$, that is, the same construction being made, $EH$ shall be less than $EK$.

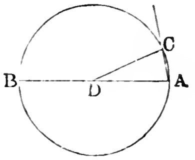
For, because $BC$ is greater than $FG$, $BH$ is greater than $FK$.
But the squares on $BH$, $HE$ are equal to the squares on $FK$, $KE$;
and the square on $BH$ is greater than the square on $FK$, because $BH$ is greater than $FK$;
therefore the square on $HE$ is less than the square on $KE$;
and therefore the straight line $EH$ is less than the straight line $EK$.

Wherefore, *the diameter* &c. Q.E.D.

## PROPOSITION 16. *THEOREM.*

*The straight line drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.*

Let $ABC$ be a circle, of which $D$ is the centre and $AB$ a diameter: the straight line drawn at right angles to $AB$, from its extremity $A$, shall fall without the circle.



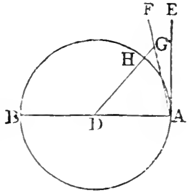

For, if not, let it fall, if possible, within the circle, as $AC$, and draw $DC$ to the point $C$, where it meets the circumference.

Then, because $DA$ is equal to $DC$, [I. *Definition* 15.
the angle $DAC$ is equal to the angle $DCA$. [I. 5.
But the angle $DAC$ is a right angle; [*Hypothesis.*
therefore the angle $DCA$ is a right angle;
and therefore the angles $DAC$, $DCA$ are equal to two right angles; which is impossible. [I. 17.
Therefore the straight line drawn from $A$ at right angles to $AB$ does not fall within the circle.

And in the same manner it may be shewn that it does not fall on the circumference.

Therefore it must fall without the circle, as $AE$.

Also between the straight line $AE$ and the circumference, no straight line can be drawn from the point $A$, which does not cut the circle.




For, if possible, let $AF$ be between them; and from the centre $D$ draw $DG$ perpendicular to $AF$; [I. 12.
let $DG$ meet the circumference at $H$.

Then, because the angle $DGA$ is a right angle, [*Construction.*
the angle $DAG$ is less than a right angle; [I. 17.
therefore $DA$ is greater than $DG$. [I. 19.
But $DA$ is equal to $DH$; [I. *Definition* 15.
therefore $DH$ is greater than $DG$, the less than the greater; which is impossible.

Therefore no straight line can be drawn from the point $A$ between $AE$ and the circumference, so as not to cut the circle.

Wherefore, *the straight line* &c. Q.E.D.

COROLLARY. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; [III. *Def.* 2.

and that it touches the circle at one point only,

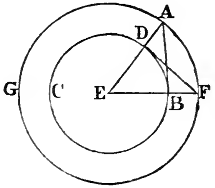
because if it did meet the circle at two points it would fall within it. [III. 2.

Also it is evident, that there can be but one straight line which touches the circle at the same point.

PROPOSITION 17. *PROBLEM.*

*To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.*

First, let the given point $A$ be without the given circle $BCD$: it is required to draw from $A$ a straight line, which shall touch the given circle.




Take $E$, the centre of the circle, [III. 1.

and join $AE$ cutting the circumference of the given circle at $D$; and from the centre $E$, at the distance $EA$, describe the circle $AFG$; from the point $D$ draw $DF$ at right angles to $EA$, [I. 11.

and join $EF$ cutting the circumference of the given circle at $B$; join $AB$. $AB$ shall touch the circle $BCD$.

For, because $E$ is the centre of the circle $AFG$, $EA$ is equal to $EF$. [I. *Definition* 15.

And because $E$ is the centre of the circle $BCD$, $EB$ is equal to $ED$. [I. *Definition* 15.

Therefore the two sides $AE$, $EB$ are equal to the two sides $FE$, $ED$, each to each;

and the angle at $E$ is common to the two triangles $AEB$, $FED$;

therefore the triangle $AEB$ is equal to the triangle $FED$, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle $ABE$ is equal to the angle $FDE$.
But the angle $FDE$ is a right angle; [*Construction.*
therefore the angle $ABE$ is a right angle. [*Axiom* 1.
And $EB$ is drawn from the centre; but the straight line drawn at right angles to a diameter of a circle, from the extremity of it, touches the circle; [III. 16, *Corollary.*
therefore $AB$ touches the circle.
And $AB$ is drawn from the given point $A$. Q.E.F.

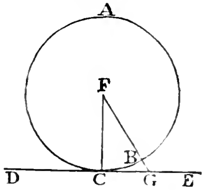
But if the given point be in the circumference of the circle, as the point $D$, draw $DE$ to the centre $E$, and $DF$ at right angles to $DE$; then $DF$ touches the circle. [III. 16, *Cor.*

PROPOSITION 18. *THEOREM.*

*If a straight line touch a circle the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.*

Let the straight line $DE$ touch the circle $ABC$ at the point $C$; take $F$, the centre of the circle $ABC$, and draw the straight line $FC$: $FC$ shall be perpendicular to $DE$.

For if not, let $FG$ be drawn from the point $F$ perpendicular to $DE$, meeting the circumference at $B$.




Then, because $FGC$ is a right angle, [*Hypothesis.*
$FCG$ is an acute angle; [I. 17.
and the greater angle of every triangle is subtended by the greater side; [I. 19.
therefore $FC$ is greater than $FG$.
But $FC$ is equal to $FB$; [I. *Definition* 15.
therefore $FB$ is greater than $FG$, the less than the greater; which is impossible.
Therefore $FG$ is not perpendicular to $DE$.

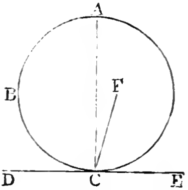
In the same manner it may be shewn that no other straight line from $F$ is perpendicular to $DE$, but $FC$; therefore $FC$ is perpendicular to $DE$.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 19. *THEOREM.*

*If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.*

Let the straight line $DE$ touch the circle $ABC$ at $C$, and from $C$ let $CA$ be drawn at right angles to $DE$: the centre of the circle shall be in $CA$.




For, if not, if possible, let $F$ be the centre, and join $CF$.

Then, because $DE$ touches the circle $ABC$, and $FC$ is drawn from the centre to the point of contact, $FC$ is perpendicular to $DE$; [III. 18.

therefore the angle $FCE$ is a right angle.

But the angle $ACE$ is also a right angle; [*Construction.*

therefore the angle $FCE$ is equal to the angle $ACE$, [*Ax.* 11. the less to the greater; which is impossible.

Therefore $F$ is not the centre of the circle $ABC$.

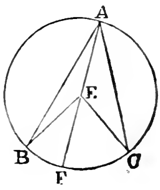
In the same manner it may be shewn that no other point out of $CA$ is the centre; therefore the centre is in $CA$.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 20. *THEOREM.*

*The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.*

Let $ABC$ be a circle, and $BEC$ an angle at the centre, and $BAC$ an angle at the circumference, which have the same arc, $BC$, for their base: the angle $BEC$ shall be double of the angle $BAC$.




Join $AE$, and produce it to $F$.

First let the centre of the circle be within the angle $BAC$.

Then, because $EA$ is equal to $EB$, the angle $EAB$ is equal to the angle $EBA$; [I. 5.

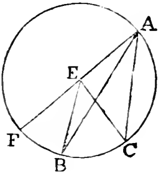
therefore the angles $EAB$, $EBA$ are double of the angle $EAB$.

But the angle $BEF$ is equal to the angles $EAB$, $EBA$; [I. 32.
therefore the angle $BEF$ is double of the angle $EAB$.

For the same reason the angle $FEC$ is double of the angle $EAC$.

Therefore the whole angle $BEC$ is double of the whole angle $BAC$.

Next, let the centre of the circle be without the angle $BAC$.



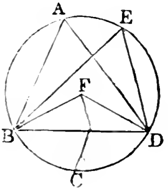

Then it may be shewn, as in the first case, that the angle $FEC$ is double of the angle $FAC$, and that the angle $FEB$, a part of the first, is double of the angle $FAB$, a part of the other; therefore the remaining angle $BEC$ is double of the remaining angle $BAC$.

Wherefore, *the angle at the centre* &c. Q.E.D.

PROPOSITION 21. *THEOREM.*

*The angles in the same segment of a circle are equal to one another.*

Let $ABCD$ be a circle, and $BAD$, $BED$ angles in the same segment $BAED$: the angles $BAD$, $BED$ shall be equal to one another.




Take $F$ the centre of the circle $ABCD$. [III. 1.

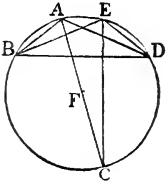
First let the segment $BAED$ be greater than a semicircle.

Join $BF$, $DF$.

Then, because the angle $BFD$ is at the centre, and the angle $BAD$ is at the circumference, and that they have the same arc for their base, namely, $BCD$;
therefore the angle $BFD$ is double of the angle $BAD$. [III. 20.
For the same reason the angle $BFD$ is double of the angle $BED$.
Therefore the angle $BAD$ is equal to the angle $BED$. [*Ax.* 7.

Next, let the segment $BAED$ be not greater than a semicircle.

Draw $AF$ to the centre, and produce it to meet the circumference at $C$, and join $CE$.




Then the segment $BAEC$ is greater than a semicircle, and therefore the angles $BAC$, $BEC$ in it, are equal, by the first case.

For the same reason, because the segment $CAED$ is greater than a semicircle, the angles $CAD$, $CED$ are equal.

Therefore the whole angle $BAD$ is equal to the whole angle $BED$. [*Axiom* 2.

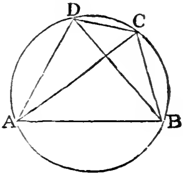
Wherefore, *the angles in the same segment* &c. Q.E.D.

## PROPOSITION 22. *THEOREM.*

*The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.*

Let $ABCD$ be a quadrilateral figure inscribed in the circle $ABCD$: any two of its opposite angles shall be together equal to two right angles.

Join $AC$, $BD$.




Then, because the three angles of every triangle are together equal to two right angles, [I. 32.

the three angles of the triangle $CAB$, namely, $CAB$, $ACB$, $ABC$ are together equal to two right angles.

But the angle $CAB$ is equal to the angle $CDB$, because they are in the same segment $CDAB$; [III. 21.

and the angle $ACB$ is equal to the angle $ADB$, because they are in the same segment $ADCB$;

therefore the two angles $CAB$, $ACB$ are together equal to the whole angle $ADC$. [*Axiom* 2.

To each of these equals add the angle $ABC$;

therefore the three angles $CAB$, $ACB$, $ABC$, are equal to the two angles $ABC$, $ADC$.

But the angles $CAB$, $ACB$, $ABC$ are together equal to two right angles; [I. 32.

therefore also the angles $ABC$, $ADC$ are together equal to two right angles.

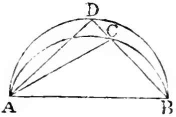
In the same manner it may be shewn that the angles $BAD$, $BCD$ are together equal to two right angles.

Wherefore, *the opposite angles* &c. Q.E.D.

PROPOSITION 23. *THEOREM.*

*On the same straight line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.*

If it be possible, on the same straight line $AB$, and on the same side of it, let there be two similar segments of circles $ACB$, $ADB$, not coinciding with one another.




Then, because the circle $ACB$ cuts the circle $ADB$ at the two points $A$, $B$, they cannot cut one another at any other point; [III. 10.

therefore one of the segments must fall within the other; let $ACB$ fall within $ADB$; draw the straight line $BCD$, and join $AC$, $AD$.

Then, because $ACB$, $ADB$ are, by hypothesis, similar segments of circles, and that similar segments of circles contain equal angles, [III. *Definition* 11.

therefore the angle $ACB$ is equal to the angle $ADB$;

that is, the exterior angle of the triangle $ACD$ is equal to the interior and opposite angle;

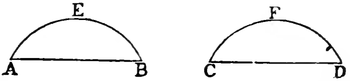
which is impossible. [I. 16.

**Wherefore,** *on the same straight line* &c. Q.E.D.

PROPOSITION 24. *THEOREM.*

*Similar segments of circles on equal straight lines are equal to one another.*

Let $AEB$, $CFD$ be similar segments of circles on the equal straight lines $AB$, $CD$: the segment $AEB$ shall be equal to the segment $CFD$.




For if the segment $AEB$ be applied to the segment $CFD$, so that the point $A$ may be on the point $C$, and the straight line $AB$ on the straight line $CD$, the point $B$ will coincide with the point $D$, because $AB$ is equal to $CD$.

Therefore, the straight line $AB$ coinciding with the straight line $CD$, the segment $AEB$ must coincide with the segment $CFD$; [III. 23.

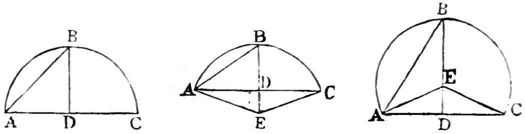
and is therefore equal to it.

Wherefore, *similar segments* &c. Q.E.D.

PROPOSITION 25. *PROBLEM.*

*A segment of a circle being given, to describe the circle of which it is a segment.*

Let $ABC$ be the given segment of a circle: it is required to describe the circle of which it is a segment.




Bisect $AC$ at $D$; [I. 10.

from the point $D$ draw $DB$ at right angles to $AC$; [I. 11.

and join $AB$.

First, let the angles $ABD$, $BAD$, be equal to one another.

Then $DB$ is equal to $DA$; [I. 6.

but $DA$ is equal to $DC$; [*Construction.*

therefore $DB$ is equal to $DC$. [*Axiom* 1.

Therefore the three straight lines $DA, DB, DC$ are all equal; and therefore $D$ is the centre of the circle. [III. 9.

From the centre $D$, at the distance of any of the three $DA$, $DB$, $DC$, describe a circle; this will pass through the other points, and the circle of which $ABC$ is a segment is described.

And because the centre $D$ is in $AC$, the segment $ABC$ is a semicircle.

Next, let the angles $ABD$, $BAD$ be not equal to one another.

At the point $A$, in the straight line $AB$, make the angle $BAE$ equal to the angle $ABD$; [I. 23.

produce $BD$, if necessary, to meet $AE$ at $E$, and join $EC$.

Then, because the angle $BAE$ is equal to the angle $ABE$, [*Construction.*

$EA$ is equal to $EB$. [I. 6.

And because $AD$ is equal to $CD$, [*Construction.*

and $DE$ is common to the two triangles $ADE$, $CDE$;

the two sides $AD$, $DE$ are equal to the two sides $CD$, $DE$, each to each;

and the angle $ADE$ is equal to the angle $CDE$, for each of them is a right angle; [*Construction.*

therefore the base $EA$ is equal to the base $EC$. [I. 4.

But $EA$ was shewn to be equal to $EB$;

therefore $EB$ is equal to $EC$. [*Axiom* 1.

Therefore the three straight lines $EA, EB, EC$ are all equal; and therefore $E$ is the centre of the circle. [III. 9.

From the centre $E$, at the distance of any of the three $EA$, $EB$, $EC$, describe a circle; this will pass through the other points, and the circle of which $ABC$ is a segment is described.

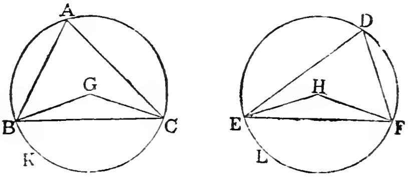
And it is evident, that if the angle $ABD$ be greater than the angle $BAD$, the centre $E$ falls without the segment $ABC$, which is therefore less than a semicircle; but if the angle $ABD$ be less than the angle $BAD$, the centre $E$ falls within the segment $ABC$, which is therefore greater than a semicircle.

Wherefore, *a segment of a circle being given, the circle has been described of which it is a segment.* Q.E.F.

PROPOSITION 26. *THEOREM.*

*In equal circles, equal angles stand on equal arcs, whether they be at the centres or circumferences.*

Let $ABC$, $DEF$ be equal circles; and let $BGC$, $EHF$ be equal angles in them at their centres, and $BAC$, $EDF$ equal angles at their circumferences: the arc $BKC$ shall be equal to the arc $ELF$.




Join $BC$, $EF$.

Then, because the circles $ABC$, $DEF$ are equal, [*Hyp.*
the straight lines from their centres are equal; [III. *Def.* 1.
therefore the two sides $BG$, $GC$ are equal to the two sides $EH$, $HF$, each to each;
and the angle at $G$ is equal to the angle at $H$; [*Hypothesis.*
therefore the base $BC$ is equal to the base $EF$. [I. 4.

And because the angle at $A$ is equal to the angle at $D$, [*Hyp.*
the segment $BAC$ is similar to the segment $EDF$; [III. *Def.* 11.
and they are on equal straight lines $BC$, $EF$.
But similar segments of circles on equal straight lines are equal to one another; [III. 24.
therefore the segment $BAC$ is equal to the segment $EDF$.

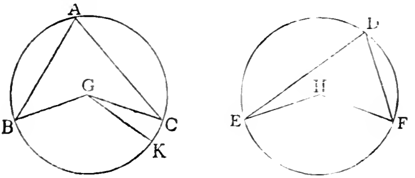
But the whole circle $ABC$ is equal to the whole circle $DEF$; [*Hypothesis.*
therefore the remaining segment $BKC$ is equal to the remaining segment $ELF$; [*Axiom* 3.
therefore the arc $BKC$ is equal to the arc $ELF$.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 27. *THEOREM.*

*In equal circles, the angles which stand on equal arcs are equal to one another, whether they be at the centres or circumferences.*

Let $ABC$, $DEF$ be equal circles, and let the angles $BGC$, $EHF$ at their centres, and the angles $BAC$, $EDF$ at their circumferences, stand on equal arcs $BC$, $EF$: the angle $BGC$ shall be equal to the angle $EHF$, and the angle $BAC$ equal to the angle $EDF$.




If the angle $BGC$ be equal to the angle $EHF$, it is manifest that the angle $BAC$ is also equal to the angle $EDF$. [III. 20, *Axiom* 7.

But, if not, one of them must be the greater. Let $BGC$ be the greater, and at the point $G$, in the straight line $BG$, make the angle $BGK$ equal to the angle $EHF$. [I. 23.

Then because the angle $BGK$ is equal to the angle $EHF$, and that in equal circles equal angles stand on equal arcs, when they are at the centres, [III. 26.

therefore the arc $BK$ is equal to the arc $EF$.

But the arc $EF$ is equal to the arc $BC$; [*Hypothesis.*

therefore the arc $BK$ is equal to the arc $BC$, [*Axiom* 1.

the less to the greater; which is impossible.

Therefore the angle $BGC$ is not unequal to the angle $EHF$, that is, it is equal to it.

And the angle at $A$ is half of the angle $BGC$, and the angle at $D$ is half of the angle $EHF$; [III. 20.

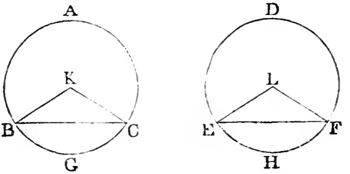
therefore the angle at $A$ is equal to the angle at $D$. [*Ax.* 7.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 28. *THEOREM.*

***In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less equal to the less.***

Let $ABC$, $DEF$ be equal circles, and $BC$, $EF$ equal straight lines in them, which cut off the two greater arcs $BAC$, $EDF$, and the two less arcs $BGC$, $EHF$: the greater arc $BAC$ shall be equal to the greater arc $EDF$, and the less arc $BGC$ equal to the less arc $EHF$.




Take $K$, $L$, the centres of the circles, [III. 1.
and join $BK$, $KC$, $EL$, $LF$.

Then, because the circles are equal, [*Hypothesis.*
the straight lines from their centres are equal; [III. *Def.* 1.
therefore the two sides $BK$, $KC$ are equal to the two sides $EL$, $LF$, each to each;
and the base $BC$ is equal to the base $EF$; [*Hypothesis.*
therefore the angle $BKC$ is equal to the angle $ELF$. [I. 8.
But in equal circles equal angles stand on equal arcs, when they are at the centres, [III. 26.
therefore the arc $BGC$ is equal to the arc $EHF$.

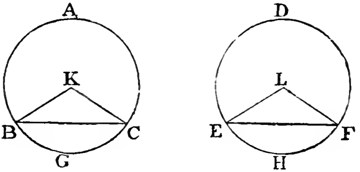
But the circumference $ABGC$ is equal to the circumference $DEHF$; [*Hypothesis.*
therefore the remaining arc $BAC$ is equal to the remaining arc $EDF$. [*Axiom* 3.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 29. *THEOREM.*

*In equal circles, equal arcs are subtended by equal straight lines.*

Let *ABC, DEF* be equal circles, and let *BGC, EHF* be equal arcs in them, and join *BC, EF*: the straight line *BC* shall be equal to the straight line *EF*.




Take *K, L*, the centres of the circles, [III. 1.
and join *BK, KC, EL, LF*.

Then, because the arc *BGC* is equal to the arc *EHF*, [*Hypothesis.*
the angle *BKC* is equal to the angle *ELF*. [III. 27.
And because the circles *ABC, DEF* are equal, [*Hypothesis.*
the straight lines from their centres are equal; [III. *Def.* 1.
therefore the two sides *BK, KC* are equal to the two sides *EL, LF*, each to each;
and they contain equal angles;
therefore the base *BC* is equal to the base *EF*. [I. 4.

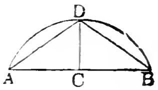
Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 30. *PROBLEM.*

*To bisect a given arc, that is, to divide it into two equal parts.*

Let *ADB* be the given arc: it is required to bisect it.

Join *AB*;
bisect it at *C*; [I. 10.
from the point *C* draw *CD* at right angles to *AB* meeting the arc at *D*. [I. 11.
The arc *ADB* shall be bisected at the point *D*.




Join *AD*, *DB*.

Then, because *AC* is equal to *CB*, [*Construction*.

and *CD* is common to the two triangles *ACD*, *BCD*:

the two sides *AC*, *CD* are equal to the two sides *BC*, *CD*, each to each;

and the angle *ACD* is equal to the angle *BCD*, because each of them is a right angle: [*Construction*.

therefore the base *AD* is equal to the base *BD*. [I. 4.

But equal straight lines cut off equal arcs, the greater equal to the greater, and the less equal to the less; [III. 28.

and each of the arcs *AD*, *DB* is less than a semi-circumference, because *DC*, if produced, is a diameter; [III. 1. *Cor*.

therefore the arc *AD* is equal to the arc *DB*.

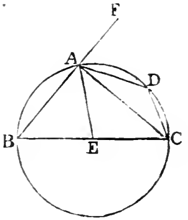
Wherefore *the given arc is bisected at D*. Q.E.F.

## PROPOSITION 31. *THEOREM*.

*In a circle the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.*

Let *ABCD* be a circle, of which *BC* is a diameter and *E* the centre; and draw *CA*, dividing the circle into the segments *ABC*, *ADC*, and join *BA*, *AD*, *DC*: the angle in the semicircle *BAC* shall be a right angle; but the angle in the segment *ABC*, which is greater than a

semicircle, shall be less than a right angle; and the angle in the segment $ADC$, which is less than a semicircle, shall be greater than a right angle.




Join $AE$, and produce $BA$ to $F$.

Then, because $EA$ is equal to $EB$, [I. *Definition* 15.

the angle $EAB$ is equal to the angle $EBA$; [I. 5.

and, because $EA$ is equal to $EC$, the angle $EAC$ is equal to the angle $ECA$;

therefore the whole angle $BAC$ is equal to the two angles, $ABC$, $ACB$. [*Axiom* 2.

But $FAC$, the exterior angle of the triangle $ABC$, is equal to the two angles $ABC$, $ACB$; [I. 32.

therefore the angle $BAC$ is equal to the angle $FAC$, [*Ax.* 1.

and therefore each of them is a right angle. [I. *Def.* 10.

Therefore the angle in a semicircle $BAC$ is a right angle.

And because the two angles $ABC$, $BAC$, of the triangle $ABC$, are together less than two right angles, [I. 17.

and that $BAC$ has been shewn to be a right angle,

therefore the angle $ABC$ is less than a right angle.

Therefore the angle in a segment $ABC$, greater than a semicircle, is less than a right angle.

And because $ABCD$ is a quadrilateral figure in a circle, any two of its opposite angles are together equal to two right angles; [III. 22.

therefore the angles $ABC$, $ADC$ are together equal to two right angles.

But the angle $ABC$ has been shewn to be less than a right angle;

therefore the angle $ADC$ is greater than a right angle.

Therefore the angle in a segment $ADC$, less than a semicircle, is greater than a right angle.

Wherefore, *the angle* &c. Q.E.D.

COROLLARY. From the demonstration it is manifest that if one angle of a triangle be equal to the other two, it is a right angle.

For the angle adjacent to it is equal to the same two angles; [I. 32.

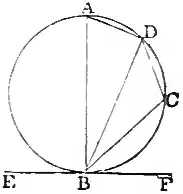
and when the adjacent angles are equal, they are right angles. [I. *Definition* 10.

## PROPOSITION 32. *THEOREM.*

*If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.*

Let the straight line $EF$ touch the circle $ABCD$ at the point $B$, and from the point $B$ let the straight line $BD$ be drawn, cutting the circle: the angles which $BD$ makes with the touching line $EF$ shall be equal to the angles in the alternate segments of the circle; that is, the angle $DBF$ shall be equal to the angle in the segment $BAD$, and the angle $DBE$ shall be equal to the angle in the segment $BCD$.

From the point $B$ draw $BA$ at right angles to $EF$, [I. 11.
and take any point $C$ in the arc $BD$, and join $AD$, $DC$, $CB$.




Then, because the straight line $EF$ touches the circle $ABCD$ at the point $B$, [*Hyp.*
and $BA$ is drawn at right angles to the touching line from the point of contact $B$, [*Construction*
therefore the centre of the circle is in $BA$. [III. 19.
Therefore the angle $ADB$, being in a semicircle, is a right angle. [III. 31.
Therefore the other two angles $BAD$, $ABD$ are equal to a right angle. [I. 32.
But $ABF$ is also a right angle [*Construction.*

Therefore the angle $ABF$ is equal to the angles $BAD$, $ABD$.
From each of these equals take away the common angle $ABD$;
therefore the remaining angle $DBF$ is equal to the remaining angle $BAD$, [Axiom 3
which is in the alternate segment of the circle.

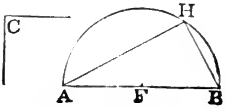
And because $ABCD$ is a quadrilateral figure in a circle, the opposite angles $BAD$, $BCD$ are together equal to two right angles. [III. 22.
But the angles $DBF$, $DBE$ are together equal to two right angles. [I. 13.
Therefore the angles $DBF$, $DBE$ are together equal to the angles $BAD$, $BCD$.
And the angle $DBF$ has been shewn equal to the angle $BAD$;
therefore the remaining angle $DBE$ is equal to the remaining angle $BCD$, [Axiom 3.
which is in the alternate segment of the circle.

Wherefore, *if a straight line* &c. Q.E.D.

## PROPOSITION 33. *PROBLEM.*

*On a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.*

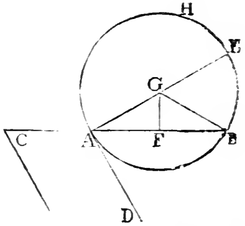
Let $AB$ be the given straight line, and $C$ the given rectilineal angle: it is required to describe, on the given straight line $AB$, a segment of a circle containing an angle equal to the angle $C$.



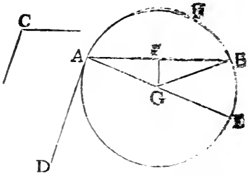

First, let the angle $C$ be a right angle.
Bisect $AB$ at $F$, [I. 10.
and from the centre $F$, at the distance $FB$, describe the semicircle $AHB$.

Then the angle $AHB$ in a semicircle is equal to the right angle $C$. [III. 31.

But if the angle $C$ be not a right angle, at the point $A$, in the straight line $AB$, make the angle $BAD$ equal to the angle $C$; [I. 23.
from the point $A$, draw $AE$ at right angles to $AD$; [I. 11.
bisect $AB$ at $F$; [I. 10.
from the point $F$, draw $FG$ at right angles to $AB$; [I. 11.
and join $GB$.




Then, because $AF$ is equal to $BF$, [Const.
and $FG$ is common to the two triangles $AFG$, $BFG$;
the two sides $AF$, $FG$ are equal to the two sides $BF$, $FG$, each to each;
and the angle $AFG$ is equal to the angle $BFG$; [I. *Definition* 10.
therefore the base $AG$ is equal to the base $BG$; [I. 4.
and therefore the circle described from the centre $G$, at the distance $GA$, will pass through the point $B$.




Let this circle be described; and let it be $AHB$.
The segment $AHB$ shall contain an angle equal to the given rectilineal angle $C$.

Because from the point $A$, the extremity of the diameter $AE$, $AD$ is drawn at right angles to $AE$, [*Construction*.
therefore $AD$ touches the circle. [III. 16. *Corollary*.
And because $AB$ is drawn from the point of contact $A$, the angle $DAB$ is equal to the angle in the alternate segment $AHB$. [III. 32.
But the angle $DAB$ is equal to the angle $C$. [*Constr*.
Therefore the angle in the segment $AHB$ is equal to the angle $C$. [*Axiom* 1.

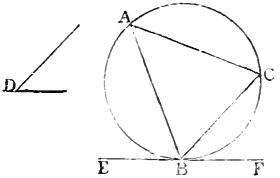
Wherefore, *on the given straight line* $AB$, *the segment* $AHB$ *of a circle has been described, containing an angle equal to the given angle* $C$. Q.E.F.

## PROPOSITION 34. *PROBLEM.*

***From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.***

Let $ABC$ be the given circle, and $D$ the given rectilineal angle: it is required to cut off from the circle $ABC$ a segment containing an angle equal to the angle $D$.

Draw the straight line $EF$ touching the circle $ABC$ at the point $B$; [III. 17.
and at the point $B$, in the straight line $BF$, make the angle $FBC$ equal to the angle $D$. [I. 23.
The segment $BAC$ shall contain an angle equal to the angle $D$.




Because the straight line $EF$ touches the circle $ABC$, and $BC$ is drawn from the point of contact $B$, [*Constr.*
therefore the angle $FBC$ is equal to the angle in the alternate segment $BAC$ of the circle. [III. 32.

But the angle $FBC$ is equal to the angle $D$. [*Construction.*

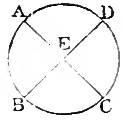
Therefore the angle in the segment $BAC$ is equal to the angle $D$. [*Axiom* 1.

Wherefore, *from the given circle $ABC$, the segment $BAC$ has been cut off, containing an angle equal to the given angle $D$.* Q.E.F.

## PROPOSITION 35. *THEOREM.*

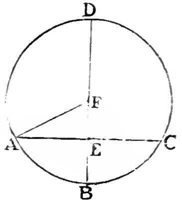
***If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.***

Let the two straight lines $AC$, $BD$ cut one another at the point $E$, within the circle $ABCD$: the rectangle contained by $AE$, $EC$ shall be equal to the rectangle contained by $BE$, $ED$.




If $AC$ and $BD$ both pass through the centre, so that $E$ is the centre, it is evident, since $EA$, $EB$, $EC$, $ED$ are all equal, that the rectangle $AE$, $EC$ is equal to the rectangle $BE$, $ED$.

But let one of them, $BD$, pass through the centre, and cut the other $AC$, which does not pass through the centre, at right angles, at the point $E$. Then, if $BD$ be bisected at $F$, $F$ is the centre of the circle $ABCD$: join $AF$.




Then, because the straight line $BD$ which passes through the centre, cuts the straight line $AC$, which does not pass through the centre, at right angles at the point $E$, [*Hypothesis.*
$AE$ is equal to $EC$. [III. 3.

And because the straight line $BD$ is divided into two equal parts at the point $F$, and into two unequal parts at the point $E$, the rectangle $BE$, $ED$, together with the square on $EF$, is equal to the square on $FB$, [II. 5.
that is, to the square on $AF$.

But the square on $AF$ is equal to the squares on $AE$, $EF$. [I. 47.

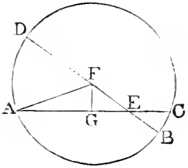
Therefore the rectangle $BE$, $ED$, together with the square on $EF$, is equal to the squares on $AE$, $EF$. [*Axiom* 1.

Take away the common square on $EF$;

then the remaining rectangle $BE$, $ED$, is equal to the remaining square on $AE$,

that is, to the rectangle $AE$, $EC$.

Next, let $BD$, which passes through the centre, cut the other $AC$, which does not pass through the centre, at the point $E$, but not at right angles. Then, if $BD$ be bisected at $F$, $F$ is the centre of the circle $ABCD$; join $AF$, and from $F$ draw $FG$ perpendicular to $AC$. [I. 12.




Then $AG$ is equal to $GC$; [III. 3.
therefore the rectangle $AE, EC$, together with the square on $EG$, is equal to the square on $AG$. [II. 5.

To each of these equals add the square on $GF$;
then the rectangle $AE, EC$, together with the squares on $EG, GF$, is equal to the squares on $AG, GF$. [*Axiom* 2.

But the squares on $EG, GF$ are equal to the square on $EF$;

and the squares on $AG, GF$ are equal to the square on $AF$. [I. 47.

Therefore the rectangle $AE, EC$, together with the square on $EF$, is equal to the square on $AF$,

that is, to the square on $FB$.

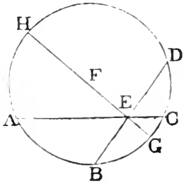
But the square on $FB$ is equal to the rectangle $BE, ED$, together with the square on $EF$. [II. 5.

Therefore the rectangle $AE, EC$, together with the square on $EF$, is equal to the rectangle $BE, ED$, together with the square on $EF$.

Take away the common square on $EF$;

then the remaining rectangle $AE, EC$ is equal to the remaining rectangle $BE, ED$. [*Axiom* 3.

Lastly, let neither of the straight lines $AC, BD$ pass through the centre.




Take the centre $F$, [III. 1.
and through $E$, the intersection of the straight lines $AC, BD$, draw the diameter $GEFH$.

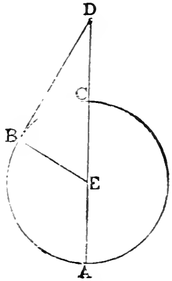
Then, as has been shewn, the rectangle $GE, EH$ is equal to the rectangle $AE, EC$, and also to the rectangle $BE, ED$;
therefore the rectangle $AE, EC$ is equal to the rectangle $BE, ED$. [*Axiom* 1.

Wherefore, *if two straight lines* &c. Q.E.D.

PROPOSITION 36. *THEOREM.*

*If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.*

Let *D* be any point without the circle *ABC*, and let *DCA*, *DB* be two straight lines drawn from it, of which *DCA* cuts the circle and *DB* touches it: the rectangle *AD*, *DC* shall be equal to the square on *DB*.




First, let *DCA* pass through the centre *E*, and join *EB*.

Then *EBD* is a right angle. [III. 18.

And because the straight line *AC* is bisected at *E*, and produced to *D*, the rectangle *AD*, *DC* together with the square on *EC* is equal to the square on *ED*. [II. 6

But *EC* is equal to *EB*;
therefore the rectangle *AD*, *DC* together with the square on *EB* is equal to the square on *ED*.

But the square on *ED* is equal to the squares on *EB*, *BD*, because *EBD* is a right angle. [I. 47.

Therefore the rectangle *AD*, *DC*, together with the square on *EB* is equal to the squares on *EB*, *BD*.

Take away the common square on *EB*;

then the remaining rectangle *AD*, *DC* is equal to the square on *DB*. [*Axiom* 3.

Next let *DCA* not pass through the centre of the circle *ABC*: take the centre *E*; [III. 1.

from *E* draw *EF* perpendicular to *AC*; [I. 12.

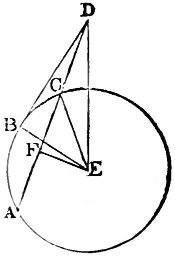
and join *EB*, *EC*, *ED*.

Then, because the straight line *EF* which passes through the centre, cuts the straight line *AC*, which does not pass through the centre, at right angles, it also bisects it; [III. 3.
therefore *AF* is equal to *FC*.

And because the straight line $AC$ is bisected at $F$, and produced to $D$, the rectangle $AD$, $DC$, together with the square on $FC$, is equal to the square on $FD$. [II. 6.

To each of these equals add the square on $FE$.

Therefore the rectangle $AD$, $DC$ together with the squares on $CF$, $FE$, is equal to the squares on $DF$, $FE$. [*Axiom* 2.




But the squares on $CF$, $FE$ are equal to the square on $CE$, because $CFE$ is a right angle; [I. 47.

and the squares on $DF$, $FE$ are equal to the square on $DE$.

Therefore the rectangle $AD$, $DC$, together with the square on $CE$, is equal to the square on $DE$.

But $CE$ is equal to $BE$;

therefore the rectangle $AD$, $DC$, together with the square on $BE$, is equal to the square on $DE$.

But the square on $DE$ is equal to the squares on $DB$, $BE$, because $EBD$ is a right angle. [I. 47.

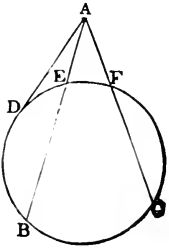
Therefore the rectangle $AD$, $DC$, together with the square on $BE$, is equal to the squares on $DB$, $BE$.

Take away the common square on $BE$;

then the remaining rectangle $AD$, $DC$ is equal to the square on $DB$. [*Axiom* 3.

Wherefore, *if from any point* &c. Q.E.D.

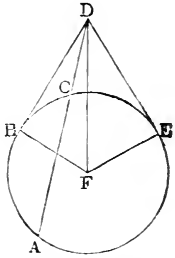
Corollary. If from any point without a circle, there be drawn two straight lines cutting it, as $AB$, $AC$, the rectangles contained by the whole lines and the parts of them without the circles are equal to one another; namely, the rectangle $BA$, $AE$ is equal to the rectangle $CA$, $AF$; for each of them is equal to the square on the straight line $AD$, which touches the circle.



PROPOSITION 37. *THEOREM.*

*If from any point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall touch it.*

Let any point $D$ be taken without the circle $ABC$, and from it let two straight lines $DCA$, $DB$ be drawn, of which $DCA$ cuts the circle, and $DB$ meets it; and let the rectangle $AD$, $DC$ be equal to the square on $DB$: $DB$ shall touch the circle.




Draw the straight line $DE$, touching the circle $ABC$; [III. 17.
find $F$ the centre, [III. 1.
and join $FB$, $FD$, $FE$.

Then the angle $FED$ is a right angle. [III. 18.
And because $DE$ touches the circle $ABC$, and $DCA$ cuts it, the rectangle $AD$, $DC$ is equal to the square on $DE$. [III. 36.
But the rectangle $AD$, $DC$ is equal to the square on $DB$. [*Hyp.*
Therefore the square on $DE$ is equal to the square on $DB$; [*Ax.* 1.
therefore the straight line $DE$ is equal to the straight line $DB$.

And $EF$ is equal to $BF$; [I. *Definition* 15.
therefore the two sides $DE$, $EF$ are equal to the two sides $DB$, $BF$ each to each;
and the base $DF$ is common to the two triangles $DEF$, $DBF$;
therefore the angle $DEF$ is equal to the angle $DBF$. [I. 8.
But $DEF$ is a right angle; [*Construction.*
therefore also $DBF$ is a right angle.
And $BF$, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter from the extremity of it touches the circle; [III. 16. *Corollary.*
therefore $DB$ touches the circle $ABC$.

Wherefore, *if from a point* &c. Q.E.D.

# EXERCISES IN EUCLID.

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## I. 1 to 15.

1. On a given straight line describe an isosceles triangle having each of the sides equal to a given straight line.

2. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

4. If the angles $ABC$ and $ACB$ at the base of an isosceles triangle be bisected by the straight lines $BD$, $CD$, shew that $DBC$ will be an isosceles triangle.

5. $BAC$ is a triangle having the angle $B$ double of the angle $A$. If $BD$ bisects the angle $B$ and meets $AC$ at $D$, shew that $BD$ is equal to $AD$.

6. In the figure of I. 5 if $FC$ and $BG$ meet at $H$ shew that $FH$ and $GH$ are equal

7. In the figure of I. 5 if $FC$ and $BG$ meet at $H$, shew that $AH$ bisects the angle $BAC$.

8. The sides $AB$, $AD$ of a quadrilateral $ABCD$ are equal, and the diagonal $AC$ bisects the angle $BAD$: shew that the sides $CB$ and $CD$ are equal, and that the diagonal $AC$ bisects the angle $BCD$.

9. $ACB$, $ADB$ are two triangles on the same side of $AB$, such that $AC$ is equal to $BD$, and $AD$ is equal to $BC$, and $AD$ and $BC$ intersect at $O$: shew that the triangle $AOB$ is isosceles.

10. The opposite angles of a rhombus are equal.

11. A diagonal of a rhombus bisects each of the angles through which it passes.

12. If two isosceles triangles are on the same base the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

13. Find a point in a given straight line such that its distances from two given points may be equal.

14. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

15. A given angle $BAC$ is bisected; if $CA$ is produced to $G$ and the angle $BAG$ bisected, the two bisecting lines are at right angles.

16. If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

## I. 16 to 26.

17. $ABC$ is a triangle and the angle $A$ is bisected by a straight line which meets $BC$ at $D$; shew that $BA$ is greater than $BD$, and $CA$ greater than $CD$.

18. In the figure of I. 17 shew that $ABC$ and $ACB$ are together less than two right angles, by joining $A$ to any point in $BC$.

19. $ABCD$ is a quadrilateral of which $AD$ is the longest side and $BC$ the shortest; shew that the angle $ABC$ is greater than the angle $ADC$, and the angle $BCD$ greater than the angle $BAD$.

20. If a straight line be drawn through $A$ one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at $F$, shew that $AF$ is greater than the diagonal of the square.

21. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two, equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

22. The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

23. The four sides of any quadrilateral are together greater than the two diagonals together.

24. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

25. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

26. If the angle $C$ of a triangle is equal to the sum of the angles $A$ and $B$, the side $AB$ is equal to twice the straight line joining $C$ to the middle point of $AB$.

27. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

28. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other.

29. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines shall be equal.

30. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

31. A straight line bisects the angle $A$ of a triangle $ABC$; from $B$ a perpendicular is drawn to this bisecting straight line, meeting it at $D$, and $BD$ is produced to meet $AC$ or $AC$ produced at $E$: shew that $BD$ is equal to $DE$.

32. $AB$, $AC$ are any two straight lines meeting at $A$: through any point $P$ draw a straight line meeting them at $E$ and $F$, such that $AE$ may be equal to $AF$.

33. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

## I. 27 to 31.

34. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

35. If two straight lines $A$ and $B$ are respectively parallel to two others $C$ and $D$, shew that the inclination of $A$ to $B$ is equal to that of $C$ to $D$.

36. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is

drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

37. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

38. If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles.

39. Find a point $B$ in a given straight line $CD$, such that if $AB$ be drawn to $B$ from a given point $A$, the angle $ABC$ will be equal to a given angle.

40. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

41. The side $BC$ of a triangle $ABC$ is produced to a point $D$; the angle $ACB$ is bisected by the straight line $CE$ which meets $AB$ at $E$. A straight line is drawn through $E$ parallel to $BC$, meeting $AC$ at $F$, and the straight line bisecting the exterior angle $ACD$ at $G$. Shew that $EF$ is equal to $FG$.

42. $AB$ is the hypotenuse of a right-angled triangle $ABC$: find a point $D$ in $AB$ such that $DB$ may be equal to the perpendicular from $D$ on $AC$.

43. $ABC$ is an isosceles triangle: find points $D$, $E$ in the equal sides $AB$. $AC$ such that $BD$, $DE$, $EC$ may all be equal.

44. A straight line drawn at right angles to $BC$ the base of an isosceles triangle $ABC$ cuts the side $AB$ at $D$ and $CA$ produced at $E$: shew that $AED$ is an isosceles triangle.

## I. 32.

45. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

46. On the sides of any triangle $ABC$ equilateral triangles $BCD$, $CAE$, $ABF$ are described, all external: shew that the straight lines $AD$, $BE$, $CF$ are all equal.

47. What is the magnitude of an angle of a regular octagon?

48. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

49. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

50. $A$ is the vertex of an isosceles triangle $ABC$, and $BA$ is produced to $D$, so that $AD$ is equal to $BA$; and $DC$ is drawn: shew that $BCD$ is a right angle.

51. $ABC$ is a triangle, and the exterior angles at $B$ and $C$ are bisected by the straight lines $BD$, $CD$ respectively, meeting at $D$: shew that the angle $BDC$ together with half the angle $BAC$ make up a right angle.

52. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

53. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

54. In the triangle $ABC$ the side $BC$ is bisected at $E$ and $AB$ at $G$; $AE$ is produced to $F$ so that $EF$ is equal to $AE$, and $CG$ is produced to $H$ so that $GH$ is equal to $CG$: shew that $FB$ and $HB$ are in one straight line.

55. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

56. $AB$, $AC$ are two straight lines given in position: it is required to find in them two points $P$ and $Q$, such that, $PQ$ being joined, $AP$ and $PQ$ may together be equal to a given straight line, and may contain an angle equal to a given angle.

57. Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

58. $AEB$, $CED$ are two straight lines intersecting at $E$; straight lines $AC$, $DB$ are drawn forming two triangles $ACE$, $BED$; the angles $ACE$, $DBE$ are bisected by the straight lines $CF$, $BF$, meeting at $F$. Shew that the angle $CFB$ is equal to half the sum of the angles $EAC$, $EDB$.

59. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

60. From the angle $A$ of a triangle $ABC$ a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at $D$; from the angle $B$ a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at $E$: shew that the straight lines which join $D$ and $E$ to the middle point of $AB$ are equal.

61. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

62. In the figure of I. 1, if $C$ and $H$ be the points of intersection of the circles, and $AB$ be produced to meet one of the circles at $K$, shew that $CHK$ is an equilateral triangle.

63. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at $D$ and $E$: shew that $DE$ is parallel to the base.

64. $AB$, $AC$ are two given straight lines, and $P$ is a given point in the former: it is required to draw through $P$ a straight line to meet $AC$ at $Q$, so that the angle $APQ$ may be three times the angle $AQP$.

65. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

66. Construct a right-angled triangle, having given the hypotenuse and the difference of the sides.

67. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on it.

68. Construct a right-angled triangle, having given the perimeter and an angle.

69. Trisect a right angle.

70. Trisect a given finite straight line.

71. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

72. Describe a triangle of given perimeter, having its angles equal to those of a given triangle.

I. 33, 34.

**73.** If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

74. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

75. No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

76. If the opposite sides of a quadrilateral are equal it is a parallelogram.

77. If the opposite angles of a quadrilateral are equal it is a parallelogram.

78. The diagonals of a parallelogram bisect each other.

79. If the diagonals of a quadrilateral bisect each other it is a parallelogram.

80. If the straight line joining two opposite angles of a parallelogram bisect the angles the four sides of the parallelogram are equal.

81. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length.

82. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

83. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

84. If the diagonals of a parallelogram are equal all its angles are equal.

85. Find a point such that the perpendiculars let fall from it on two given straight lines shall be respectively equal to two given straight lines. How many such points are there?

86. It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

87. On the sides $AB$, $BC$, and $CD$ of a parallelogram $ABCD$ three equilateral triangles are described, that on $BC$ towards the same parts as the parallelogram, and those on $AB$, $CD$ towards the opposite parts: shew that the

distances of the vertices of the triangles on $AB$, $CD$ from that on $BC$ are respectively equal to the two diagonals of the parallelogram.

88. If the angle between two adjacent sides of a parallelogram be increased, while their lengths do not alter, the diagonal through their point of intersection will diminish.

89. $A$, $B$, $C$ are three points in a straight line, such that $AB$ is equal to $BC$: shew that the sum of the perpendiculars from $A$ and $C$ on any straight line which does not pass between $A$ and $C$ is double the perpendicular from $B$ on the same straight line.

90. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

91. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

92. $AB$, $AC$ are two given straight lines; through a given point $E$ between them it is required to draw a straight line $GEH$ such that the intercepted portion $GH$ shall be bisected at the point $E$.

93. Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

94. $ABCD$ is a parallelogram, and $E$, $F$, the middle points of $AD$ and $BC$ respectively; shew that $BE$ and $DF$ will trisect the diagonal $AC$.

## I. 35 to 45.

95. $ABCD$ is a quadrilateral having $BC$ parallel to $AD$; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of $DC$ a straight line parallel to $AB$.

96. $ABCD$ is a quadrilateral having $BC$ parallel to $AD$, $E$ is the middle point of $DC$; shew that the triangle $AEB$ is half the quadrilateral.

97. Shew that any straight line passing through the middle point of the diameter of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

98. Bisect a parallelogram by a straight line drawn through a given point within it.

99. Construct a rhombus equal to a given parallelogram.

100. If two triangles have two sides of the one equal to two sides of the other, each to each, and the sum of the two angles contained by these sides equal to two right angles, the triangles are equal in area.

101. A straight line is drawn bisecting a parallelogram $ABCD$ and meeting $AD$ at $E$ and $BC$ at $F$: shew that the triangles $EBF$ and $CED$ are equal.

102. Shew that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

103. Two straight lines $AB$ and $CD$ intersect at $E$, and the triangle $AEC$ is equal to the triangle $BED$: shew that $BC$ is parallel to $AD$.

104. $ABCD$ is a parallelogram; from any point $P$ in the diagonal $BD$ the straight lines $PA$, $PC$ are drawn. Shew that the triangles $PAB$ and $PCB$ are equal.

105. If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

106. The straight line which joins the middle points of two sides of any triangle is parallel to the base.

107. Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram.

108. $D$, $E$ are the middle points of the sides $AB$, $AC$ of a triangle, and $CD$, $BE$ intersect at $F$: shew that the triangle $BFC$ is equal to the quadrilateral $ADFE$.

109. The straight line which bisects two sides of any triangle is half the base.

110. In the base $AC$ of a triangle take any point $D$: bisect $AD$, $DC$, $AB$, $BC$ at the points $E$, $F$, $G$, $H$ respectively: shew that $EG$ is equal and parallel to $FH$.

111. Given the middle points of the sides of a triangle, construct the triangle.

112. If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

113. The sides $AB$, $AC$ of a given triangle $ABC$ are bisected at the points $E$, $F$; a perpendicular is drawn from $A$ to the opposite side, meeting it at $D$. Shew that the

angle $FDE$ is equal to the angle $BAC$. Shew also that $FDE$ is half the triangle $ABC$.

**114.** Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced.

115. Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

116. $ABCD$ is a parallelogram; from $D$ draw any straight line $DFG$ meeting $BC$ at $F$ and $AB$ produced at $G$: draw $AF$ and $CG$: shew that the triangles $ABF$, $CFG$ are equal.

117. $ABC$ is a given triangle: construct a triangle of equal area, having for its base a given straight line $AD$, coinciding in position with $AB$.

118. $ABC$ is a given triangle: construct a triangle of equal area, having its vertex at a given point in $BC$ and its base in the same straight line as $AB$.

119. $ABCD$ is a given quadrilateral: construct another quadrilateral of equal area having $AB$ for one side, and for another a straight line drawn through a given point in $CD$ parallel to $AB$.

120. $ABCD$ is a quadrilateral: construct a triangle whose base shall be in the same straight line as $AB$, vertex at a given point $P$ in $CD$, and area equal to that of the given quadrilateral.

121. $ABC$ is a given triangle: construct a triangle of equal area, having its base in the same straight line as $AB$, and its vertex in a given straight line parallel to $AB$.

122. Bisect a given triangle by a straight line drawn through a given point in a side.

123. Bisect a given quadrilateral by a straight line drawn through a given angular point.

124. If through the point $O$ within a parallelogram $ABCD$ two straight lines are drawn parallel to the sides, and the parallelograms $OB$ and $OD$ are equal, the point $O$ is in the diagonal $AC$.

## I. 46 to 48.

125. On the sides $AC$, $BC$ of a triangle $ABC$, squares $ACDE$, $BCFH$ are described: shew that the straight lines $AF$ and $BD$ are equal.

126. The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

127. The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

128. If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

129. A straight line is drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line drawn parallel to it

130. If any point $P$ be joined to $A$. $B$, $C$, $D$, the angular points of a rectangle, the squares on $PA$ and $PC$ are together equal to the squares on $PB$ and $PD$.

131. In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

132. If $ABC$ be a triangle whose angle $A$ is a right angle, and $BE$, $CF$ be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on $BE$ and $CF$ is equal to five times the square on $BC$.

133. On the hypotenuse $BC$, and the sides $CA$, $AB$ of a right-angled triangle $ABC$, squares $BDEC$, $AF$, and $AG$ are described: shew that the squares on $DG$ and $EF$ are together equal to five times the square on $BC$.

## II. 1 to 11.

134. A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

135. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

136. Construct a rectangle equal to the difference of two given squares.

137. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

138. Shew that the square on the sum of two straight lines together with the square on their difference is double the squares on the two straight lines.

139. Divide a given straight line into two parts such that the sum of their squares shall be equal to a given square.

140. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.

141. In the figure of II. 11 if $CH$ be produced to meet $BF$ at $L$, shew that $CL$ is at right angles to $BF$.

142. In the figure of II. 11 if $BE$ and $CH$ meet at $O$, shew that $AO$ is at right angles to $CH$.

143. Shew that in a straight line divided as in II. 11 the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

## II. 12 to 14.

144. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on it from the opposite angle and the extremity of the base.

145. In any triangle the sum of the squares on the sides is equal to twice the square on half the base together with twice the square on the straight line drawn from the vertex to the middle point of the base.

146. *ABC* is a triangle having the sides *AB* and *AC* equal; if *AB* is produced beyond the base to *D* so that *BD* is equal to *AB*, shew that the square on *CD* is equal to the square on *AB*, together with twice the square on *BC*.

147. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

148. The base of a triangle is given and is bisected by the centre of a given circle: if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

149. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

150. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

151. The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

152. In *AB* the diameter of a circle take two points *C* and *D* equally distant from the centre, and from any point *E* in the circumference draw *EC*, *ED*: shew that the squares on *EC* and *ED* are together equal to the squares on *AC* and *AD*.

153. In *BC* the base of a triangle take *D* such that the squares on *AB* and *BD* are together equal to the squares on *AC* and *CD*, then the middle point of *AD* will be equally distant from *B* and *C*.

154. The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on a side of the triangle by the rectangle contained by the segments of the base.

155. A square *BDEC* is described on the hypotenuse *BC* of a right-angled triangle *ABC*: shew that the squares on *DA* and *AC* are together equal to the squares on *EA* and *AB*.

156. *ABC* is a triangle in which *C* is a right angle, and *DE* is drawn from a point *D* in *AC* perpendicular to

*AB*: shew that the rectangle $AB$, $AE$ is equal to the rectangle $AC$, $AD$.

157. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

158. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on this perpendicular is equal to the rectangle contained by the segments of the base.

159. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

160. In a triangle $ABC$ the angles $B$ and $C$ are acute: if $E$ and $F$ be the points where perpendiculars from the opposite angles meet the sides $AC$, $AB$, shew that the square on $BC$ is equal to the rectangle $AB$, $BF$, together with the rectangle $AC$, $CE$.

161. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

## III. 1 to 15.

162. Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

163. Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

164. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circles are equal.

165. Two circles whose centres are $A$ and $B$ intersect at $C$; through $C$ two chords $DCE$ and $FCG$ are drawn equally inclined to $AB$ and terminated by the circles: shew that $DE$ and $FG$ are equal.

166. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

167. If from any point in the diameter of a circle straight lines are drawn to the extremities of a parallel chord, the squares on these straight lines are together equal to the squares on the segments into which the diameter is divided.

168. $A$ and $B$ are two fixed points without a circle $PQR$; it is required to find a point $P$ in the circumference, so that the sum of the squares described on $AP$ and $BP$ may be the least possible.

169. If in any two given circles which touch one another, there be drawn two parallel diameters, an extremity of each diameter, and the point of contact, shall lie in the same straight line.

170. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Shew that these two chords have the segments of the one equal to the segments of the other, each to each.

171. Through a given point within a circle draw the shortest chord

172. $O$ is the centre of a circle, $P$ is any point in its circumference, $PN$ a perpendicular on a fixed diameter: shew that the straight line which bisects the angle $OPN$ always passes through one or the other of two fixed points.

173. Three circles touch one another externally at the points $A$, $B$, $C$; from $A$, the straight lines $AB$, $AC$ are produced to cut the circle $BC$ at $D$ and $E$: shew that $DE$ is a diameter of $BC$, and is parallel to the straight line joining the centres of the other circles.

174. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of any adjacent two is parallel to the common chord of the other two.

175. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

III. 16 to 19.

176. Shew that two tangents can be drawn to a circle from a given external point, and that they are of equal length.

177. Draw parallel to a given straight line a straight line to touch a given circle.

178. Draw perpendicular to a given straight line a straight line to touch a given circle.

179. In the diameter of a circle produced, determine a point so that the tangent drawn from it to the circumference shall be of given length.

180. Two circles have the same centre: shew that all chords of the outer circle which touch the inner circle are equal.

181. Through a given point draw a straight line so that the part intercepted by the circumference of a given circle shall be equal to a given straight line not greater than the diameter.

182. Two tangents are drawn to a circle at the opposite extremities of a diameter, and cut off from a third tangent a portion $AB$: if $C$ be the centre of the circle shew that $ACB$ is a right angle.

183. Describe a circle that shall have a given radius and touch a given circle and a given straight line.

184. A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle.

185. Draw a straight line to touch each of two given circles.

186. Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

187. Draw a straight line cutting two given circles so that the chords intercepted within the circles shall have given lengths.

188. A quadrilateral is described so that its sides touch a circle: shew that two of its sides are together equal to the other two sides.

189. Shew that no parallelogram can be described about a circle except a rhombus.

190. $ABD$, $ACE$ are two straight lines touching a circle at $B$ and $C$, and if $DE$ be joined $DE$ is equal to $BD$ and $CE$ together: shew that $DE$ touches the circle.

191. If a quadrilateral be described about a circle the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

192. Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

193. A straight line is drawn touching two circles: shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

194. If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

195. $AB$ is the diameter and $C$ the centre of a semicircle: shew that $O$ the centre of any circle inscribed in the semicircle is equidistant from $C$ and from the tangent to the semicircle parallel to $AB$.

196. If from any point without a circle straight lines be drawn touching it, the angle contained by the tangents is double the angle contained by the straight line joining the points of contact and the diameter drawn through one of them.

197. A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

198. If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

199. A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

200. Of all straight lines which can be drawn from two given points to meet in the convex circumference of a

given circle, the sum of the two is least which make equal angles with the tangent at the point of concourse.

201. $C$ is the centre of a given circle, $CA$ a radius, $B$ a point on a radius at right angles to $CA$; join $AB$ and produce it to meet the circle again at $D$, and let the tangent at $D$ meet $CB$ produced at $E$: shew that $BDE$ is an isosceles triangle.

202. Let the diameter $BA$ of a circle be produced to $P$, so that $AP$ equals the radius; through $A$ draw the tangent $AED$, and from $P$ draw $PEC$ touching the circle at $C$ and meeting the former tangent at $E$; join $BC$ and produce it to meet $AED$ at $D$: then will the triangle $DEC$ be equilateral.

## III. 20 to 22.

203. Two tangents $AB$, $AC$ are drawn to a circle; $D$ is any point on the circumference outside of the triangle $ABC$: shew that the sum of the angles $ABD$ and $ACD$ is constant.

204. $P$, $Q$ are any points in the circumferences of two segments described on the same straight line $AB$, and on the same side of it; the angles $PAQ$, $PBQ$ are bisected by the straight lines $AR$, $BR$ meeting at $R$: shew that the angle $ARB$ is constant.

205. Two segments of a circle are on the same base $AB$, and $P$ is any point in the circumference of one of the segments; the straight lines $APD$, $BPC$ are drawn meeting the circumference of the other segment at $D$ and $C$; $AC$ and $BD$ are drawn intersecting at $Q$. Shew that the angle $AQB$ is constant.

206. $APB$ is a fixed chord passing through $P$ a point of intersection of two circles $AQP$, $PBR$; and $QPR$ is any other chord of the circles passing through $P$: shew that $AQ$ and $RB$ when produced meet at a constant angle.

207. $AOB$ is a triangle; $C$ and $D$ are points in $BO$ and $AO$ respectively, such that the angle $ODC$ is equal to the angle $OBA$: shew that a circle may be described round the quadrilateral $ABCD$.

208. $ABCD$ is a quadrilateral inscribed in a circle, and the sides $AB$, $CD$ when produced meet at $O$: shew that the triangles $AOC$, $BOD$ are equiangular.

209. Shew that no parallelogram except a rectangle can be inscribed in a circle.

210. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

211. A quadrilateral is inscribed in a circle: shew that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

212. Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

213. Divide a circle into two parts so that the angle contained in one segment shall be equal to five times the angle contained in the other.

214. If the angle contained by any side of a quadrilateral and the adjacent side produced, be equal to the opposite angle of the quadrilateral, shew that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

215. If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

216. $A, B, C, D$ are four points taken in order on the circumference of a circle; the straight lines $AB$, $CD$ produced intersect at $P$, and $AD$, $BC$ at $Q$: shew that the straight lines which respectively bisect the angles $APC$, $AQC$ are perpendicular to each other.

217. If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

218. A quadrilateral can have one circle inscribed in it and another circumscribed about it: shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

## III. 23 to 30.

219. The straight lines joining the extremities of the chords of two equal arcs of a circle, towards the same parts are parallel to each other.

220. The straight lines in a circle which join the extremities of two parallel chords are equal to each other.

221. $AB$ is a common chord of two circles; through $C$ any point of one circumference straight lines $CAD$, $CBE$ are drawn terminated by the other circumference: shew that the arc $DE$ is invariable.

222. Through a point $C$ in the circumference of a circle two straight lines $ACB$, $DCE$ are drawn cutting the circle at $B$ and $E$: shew that the straight line which bisects the angles $ACE$, $DCB$ meets the circle at a point equidistant from $B$ and $E$.

223. The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle.

224. $AB$ is a diameter of a circle, and $D$ is a given point on the circumference, such that the arc $DB$ is less than half the arc $DA$: draw a chord $DE$ on one side of $AB$ so that the arc $EA$ may be three times the arc $BD$.

225. From $A$ and $B$ two of the angular points of a triangle $ABC$, straight lines are drawn so as to meet the opposite sides at $P$ and $Q$ in given equal angles: shew that the straight line joining $P$ and $Q$ will be of the same length in all triangles on the same base $AB$, and having vertical angles equal to $C$.

226. If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

227. $OA$, $OB$, $OC$ are three chords of a circle; the angle $AOB$ is equal to the angle $BOC$, and $OA$ is nearer to the centre than $OB$. From $B$ a perpendicular is drawn on $OA$, meeting it at $P$, and a perpendicular on $OC$ produced, meeting it at $Q$: shew that $AP$ is equal to $CQ$.

228. $AB$ is a given finite straight line; through $A$ two indefinite straight lines are drawn equally inclined to $AB$; any circle passing through $A$ and $B$ meets these straight lines at $L$ and $M$. Shew that if $AB$ be between $AL$ and $AM$ the sum of $AL$ and $AM$ is constant; if $AB$ be not between $AL$ and $AM$ the difference of $AL$ and $AM$ is constant.

229. $AOB$ and $COD$ are diameters of a circle at right angles to each other; $E$ is a point in the arc $AC$, and $EFG$ is a chord meeting $COD$ at $F$, and drawn in such a

direction that $EF$ is equal to the radius. Shew that the arc $BG$ is equal to three times the arc $AE$.

230. The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point.

231. If two circles touch each other internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

## III. 31.

232. Right-angled triangles are described on the same hypotenuse: shew that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

233. The circles described on the equal sides of an isosceles triangle as diameters, will intersect at the middle point of the base.

234. The greatest rectangle which can be inscribed in a circle is a square.

235. The hypotenuse $AB$ of a right-angled triangle $ABC$ is bisected at $D$, and $EDF$ is drawn at right angles to $AB$, and $DE$ and $DF$ are cut off each equal to $DA$; $CE$ and $CF$ are joined: shew that the last two straight lines will bisect the angle $C$ and its supplement respectively.

236. On the side $AB$ of any triangle $ABC$ as diameter a circle is described; $EF$ is a diameter parallel to $BC$: shew that the straight lines $EB$ and $FB$ bisect the interior and exterior angles at $B$.

237. If $AD$, $CE$ be drawn perpendicular to the sides $BC$, $AB$ of a triangle $ABC$, and $DE$ be joined, shew that the angles $ADE$ and $ACE$ are equal to each other.

238. If two circles $ABC$, $ABD$ intersect at $A$ and $B$, and $AC$, $AD$ be two diameters, shew that the straight line $CD$ will pass through $B$.

239. If $O$ be the centre of a circle and $OA$ a radius and a circle be described on $OA$ as diameter, the circum-

ference of this circle will bisect any chord drawn through it from $A$ to meet the exterior circle.

240. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

241. Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

242. If from the angles at the base of any triangle perpendiculars are drawn to the opposite sides, produced if necessary, the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the centre of the base.

243. $AD$ is a diameter of a circle; $B$ and $C$ are points on the circumference on the same side of $AD$; a perpendicular from $D$ on $BC$ produced through $C$, meets it at $E$: shew that the square on $AD$ is greater than the sum of the squares on $AB$, $BC$, $CD$, by twice the rectangle $BC$, $CE$.

244. $AB$ is the diameter of a semicircle, $P$ is a point on the circumference, $PM$ is perpendicular to $AB$; on $AM$, $BM$ as diameters two semicircles are described, and $AP$, $BP$ meet these latter circumferences at $Q$, $R$: shew that $QR$ will be a common tangent to them.

245. $AB$, $AC$ are two straight lines, $B$ and $C$ are given points in the same; $BD$ is drawn perpendicular to $AC$, and $DE$ perpendicular to $AB$; in like manner $CF$ is drawn perpendicular to $AB$, and $FG$ to $AC$. Shew that $EG$ is parallel to $BC$.

246. Two circles intersect at the points $A$ and $B$, from which are drawn chords to a point $C$ in one of the circumferences, and these chords, produced if necessary, cut the other circumference at $D$ and $E$: shew that the straight line $DE$ cuts at right angles that diameter of the circle $ABC$ which passes through $C$.

247. If squares be described on the sides and hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

248. $C$ is the centre of a given circle, $CA$ a straight line less than the radius; find the point of the circumference at which $CA$ subtends the greatest angle.

249. $AB$ is the diameter of a semicircle, $D$ and $E$ are any two points in its circumference. Shew that if the chords joining $A$ and $B$ with $D$ and $E$ each way intersect at $F$ and $G$, then $FG$ produced is at right angles to $AB$.

250. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: shew that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

251. A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: shew that the parallelogram thus formed is a rhombus with angles equal to those of the first.

252. If two chords of a circle meet at a right angle within or without a circle, the squares on their segments are together equal to the squares on the diameter.

## III. 32 to 34.

253. $B$ is a point in the circumference of a circle, whose centre is $C$; $PA$, a tangent at any point $P$, meets $CB$ produced at $A$, and $PD$ is drawn perpendicular to $CB$: shew that the straight line $PB$ bisects the angle $APD$.

254. If two circles touch each other, any straight line drawn through the point of contact will cut off similar segments.

255. $AB$ is any chord, and $AD$ is a tangent to a circle at $A$. $DPQ$ is any straight line parallel to $AB$, meeting the circumference at $P$ and $Q$. Shew that the triangle $PAD$ is equiangular to the triangle $QAB$.

256. Two circles $ABDH$, $ABG$, intersect each other at the points $A$, $B$; from $B$ a straight line $BD$ is drawn in the one to touch the other; and from $A$ any chord whatever is drawn cutting the circles at $G$ and $H$: shew that $BG$ is parallel to $DH$.

257. Two circles intersect at $A$ and $B$. At $A$ the tangents $AC$, $AD$ are drawn to each circle and terminated

by the circumference of the other. If $CB$, $BD$ be joined, shew that $AB$ or $AB$ produced, if necessary, bisects the angle $CBD$.

258. Two circles intersect at $A$ and $B$, and through $P$ any point in the circumference of one of them the chords $PA$ and $PB$ are drawn to cut the other circle at $C$ and $D$: shew that $CD$ is parallel to the tangent at $P$.

259. If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped on them from the middle point of the subtended arc are equal to one another.

260. $AB$ is any chord of a circle, $P$ any point on the circumference of the circle; $PM$ is a perpendicular on $AB$ and is produced to meet the circle at $Q$; and $AN$ is drawn perpendicular to the tangent at $P$: shew that the triangle $NAM$ is equiangular to the triangle $PAQ$.

261. Two diameters $AOB$, $COD$ of a circle are at right angles to each other; $P$ is a point in the circumference; the tangent at $P$ meets $COD$ produced at $Q$, and $AP$, $BP$ meet the same line at $R$, $S$ respectively: shew that $RQ$ is equal to $SQ$

262. Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

263. Construct a triangle, having given the base, the vertical angle, and the altitude.

264. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

265. Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

266. From a given point $A$ without a circle whose centre is $O$ draw a straight line cutting the circle at the points $B$ and $C$, so that the area $BOC$ may be the greatest possible.

267. Two straight lines containing a constant angle always pass through two fixed points, their position being otherwise unrestricted: shew that the straight line bisecting the angle always passes through one or other of two fixed points.

268. Given one angle of a triangle, the side opposite

it, and the sum of the other two sides, construct the triangle.

III. 35 to 37.

269. If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

270. Two circles intersect at $A$ and $B$: shew that $AB$ produced bisects their common tangent.

271. If $AD$, $CE$ are drawn perpendicular to the sides $BC$, $AB$ of a triangle $ABC$, shew that the rectangle contained by $BC$ and $BD$ is equal to the rectangle contained by $BA$ and $BE$.

272. If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

273. From a given point as centre describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

274. Two circles $ABCD$, $EBCF$, having the common tangents $AE$ and $DF$, cut one another at $B$ and $C$, and the chord $BC$ is produced to cut the tangents at $G$ and $H$: shew that the square on $GH$ exceeds the square on $AE$ or $DF$ by the square on $BC$.

275. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal: shew that the straight lines joining the two points of intersection of each pair will pass through this point.

276. $ABC$ is a right-angled triangle; from any point $D$ in the hypotenuse $BC$ a straight line is drawn at right angles to $BC$, meeting $CA$ at $E$ and $BA$ produced at $F$: shew that the square on $DE$ is equal to the difference of the rectangles $BD$, $DC$ and $AE$, $EC$; and that the square on $DF$ is equal to the sum of the rectangles $BD$, $DC$ and $AF$, $FB$.

277. It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained

by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

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